A Judgmental Analysis of Linear Logic

Bor-Yuh Evan Chang Kaustuv Chaudhuri Frank Pfenning

April 14, 2003
CMU-CS-03-131

School of Computer Science
Carnegie Mellon University
Pittsburgh, PA 15213

Abstract

We reexamine the foundations of linear logic, developing a system of natural deduction following Martin-Löf's separation of judgments from propositions. Our construction yields a clean and elegant formulation that accounts for a rich set of multiplicative, additive, and exponential connectives, extending dual intuitionistic linear logic but differing from both classical linear logic and Hyland and de Paiva's full intuitionistic linear logic. We also provide a corresponding sequent calculus that admits a simple proof of the admissibility of cut by a single structural induction. Finally, we show how to interpret classical linear logic (with or without the MIX rule) in our system, employing a form of double-negation translation.

This work has been supported by NSF Grant CCR-9988281. Bor-Yuh Evan Chang is currently at the University of California, Berkeley, CA and holds a California Microelectronics Fellowship.

Keywords: Constructive Logic, Linear Logic, Judgmental Foundations

1 Introduction

Central to the design of linear logic [13] are the beautiful symmetries exhibited by the classical sequent calculus. This has led to applications in the areas of concurrent computation and games, among others, where the symmetry captures a related symmetry in the domain. However, in many situations, the asymmetry of intuitionistic natural deduction seems a better fit. For example, functional computation has an asymmetry between a function's arguments and the value it returns. Logic programming maintains an asymmetry between the program and the goal. Intuitionistic versions of linear logic have been used to explore interesting phenomena in functional computation (see, for example, [18, 1, 6, 28, 15, 2, 5]), logic programming [14], and logical frameworks [9].

In this paper, we analyze linear logic in an inherently asymmetric natural deduction formulation following Martin-Löf's methodology of separating judgments from propositions [19]. We require minimal judgmental notions — linear hypothetical judgments, categorical judgments, and ordinary hypothetical judgments suffice to explain a full range of intuitionistic linear propositional connectives: $\otimes$, $\mathbf{1}$, $\multimap$, $\&$, $\top$, $\oplus$, $\mathbf{0}$, $!$, $?$, $\perp$, $\neg$, and $⅋$. The judgmental construction gives a clean and elegant proof theory, both in natural deduction and the sequent calculus. For example, we obtain proofs of cut-elimination by a simple structural induction. We refer to the resulting system as *judgmental intuitionistic linear logic* (JILL).

As expected, the meaning of some of the connectives in JILL differs somewhat from that in classical linear logic. Nonetheless, we do not sacrifice any expressive power because we can interpret classical linear logic (with and without the MIX rule) into a fragment of JILL. Our embedding proceeds in a compositional manner using a form of double-negation translation. Reformulating this translation can give a direct judgmental account of classical linear logic, showing that reasoning in classical linear logic corresponds to finding a contradication among assumptions about truth and falsehood, and reasoning in classical linear logic with MIX corresponds to finding a means to consume all resources and pay off all debts.

Much related work exists, so we only briefly touch upon it here. We view the judgmental reconstruction of modal logic and lax logic [23] as motivating our approach. We also owe much to Polakow's development of ordered logic [24], which employs linear and ordered hypothetical judgments, but does not introduce possibility and related connectives. JILL contains, as fragments, both dual intuitionistic linear logic (DILL) [4] and hereditary Harrop logic underlying linear logic programming [14]. The contribution of JILL with respect to these systems is the judgmental account, which gives rise to the new $?$, $\perp$, $\neg$, and $⅋$ connectives. Full intuitionistic linear logic (FILL) [16] does not have additives[1] and does not proceed via a judgmental account. It requires either proof terms [7] or occurrence labels [8] in order to formulate the rules for linear implication, which makes it difficult to understand the meanings of the connectives in isolation. On the other hand, multiplicative disjunction $⅋$ in FILL seems closer to its classical counterpart; furthermore, FILL has a clear categorical semantics [10] that we have not yet explored for JILL.

Related structural proofs of cut-elimination have appeared for intuitionistic and classical logic [22], classical linear logic in an unpublished note [21], and ordered logic [25], but these did not incorporate possibility and related connectives ($?$, $\perp$, $\neg$, and $⅋$). To our knowledge, the double-negation translation from classical into intuitionistic linear logic that can optionally account for MIX is also new in this paper. Lamarche [17] has previously given a more complex double-negation translation from classical linear logic into intuitionistic linear logic using a one-sided sequent calculus with polarities.

The remainder of the paper has the following organization. In Sec. 2, we describe natural deduction for JILL in terms of the required judgmental notions. In particular, we introduce the possibility judgment (2.3), multiplicative contradiction and negation (2.4), and the modal disjunction $⅋$ (2.5). In Sec. 3, we provide the sequent calculus for JILL and prove a structural cut

[1] These were deemed to be straightforward, although this is not obvious to the present authors.

admissibility theorem (Thm. 2). In Sec. 4, we give an interpretation of classical linear logic into JILL and further show how to modify it to give a logical justifiction for the classical MIX rule (4.2).

2 Natural Deduction for JILL

We take a foundational view of logic based on the approach laid out by Martin-Löf in his Siena lectures in 1983 [19], more recently extended to incorporate categorical judgments [23]. This view separates *judgments* from *propositions*. Evident judgments become objects of knowledge and proofs provide the requisite evidence. In logic, we concern ourselves with particular judgments such as A *is a proposition* and A *is true* (for propositions A). To understand the meaning of a proposition, we need to understand what counts as a verification of that proposition. Consequently, the inference rules characterizing truth of propositions define their meaning, as long as they satisfy certain consistency conditions that we call *local soundness* and *local completeness* [23]. We sharpen this analysis further when we introduce the sequent calculus in Sec. 3.

2.1 Linear Hypothetical Judgments

Before we describe any particular logical connective, we need to discuss the basic judgment forms. Because we focus on propositional logic, we dispense with the formation judgment "A *is a proposition*". For propositions A, we write $A\ \ true$ to express the judgment "A *is true*". Reasoning from assumptions is fundamental in logic and captured by Martin-Löf in the form of hypothetical judgments. In linear logic, we refine this notion by restricting assumptions to have *exactly one use* in a proof, resulting in the notion of a *linear hypothetical judgment*.

$$\underbrace{B_1\ \ true, \ldots, B_k\ \ true}_{\Delta} \Vdash C\ \ true$$

We refer to Δ as the *linear hypotheses*. We allow free exchange among the linear hypotheses but neither weakening nor contraction. The restriction that each linear hypothesis in Δ must be used exactly once in the proof of $C\ \ true$ suggests a view of elements of Δ as *resources* and $C\ \ true$ as a *goal* to be accomplished with the resources. A proof then corresponds to a plan for achieving the goal with the given resources. This interpretation yields the following hypothesis rule.

$$\frac{}{A\ \ true \Vdash A\ \ true}\ \mathsf{hyp}$$

Dual to the hypothesis rule, we have a principle of substitution that allows us to substitute a proof for uses of a linear hypothesis in a derivation.

Principle 1 (Substitution).
Whenever $\Delta \Vdash A\ \ true$ and $\Delta', A\ \ true \Vdash C\ \ true$, also $\Delta, \Delta' \Vdash C\ \ true$.

We do not realize the substitution principle as an inference rule in the logic but intend it as a structural property of the logic maintained by all other inference rules. In the end, once we fix a set of connectives with their inference rules, we can prove the substitution principle as a meta-theorem by induction over the structure of derivations. Meanwhile, we use the substitution principle to show local soundness and completeness of the rules characterizing the connectives.

In the spirit of the judgmental approach, we now think of the meaning of a connective as defined by its *introduction rule*. For example, $A \otimes B$ expresses the *simultaneous truth* of both A and B. In order to achieve such a goal, we have to show how to divide up our resources, devoting one portion to achieving A and the other to achieving B.

$$\frac{\Delta \Vdash A\ \ true \quad \Delta' \Vdash B\ \ true}{\Delta, \Delta' \Vdash A \otimes B\ \ true}\ \otimes I$$

Conversely, if we know $A \otimes B$ *true*, then we may assume resources A *true* and B *true*.

$$\frac{\Delta \Vdash A \otimes B\ \mathit{true} \quad \Delta', A\ \mathit{true}, B\ \mathit{true} \Vdash C\ \mathit{true}}{\Delta, \Delta' \Vdash C\ \mathit{true}} \otimes E$$

Local soundness ensures that the elimination rules for connectives are not too strong: if we have sufficient evidence for the premisses, we have sufficient evidence for the conclusion. We show this by means of a local reduction, reducing any proof containing an introduction of a connective immediately followed by its elimination to a proof without the connective. For $\otimes$, we require the substitution principle twice, substituting $\mathcal{D}_1$ for uses of A and $\mathcal{D}_2$ for uses of B in $\mathcal{E}$ to yield $\mathcal{E}'$.

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Delta \Vdash A\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Delta' \Vdash B\end{array}}{\Delta, \Delta' \Vdash A \otimes B} \otimes I \qquad \begin{array}{c}\mathcal{E}\\ \Delta'', A, B \Vdash C\end{array}}{\Delta, \Delta', \Delta'' \Vdash C} \otimes E \quad \overset{R}{\Longrightarrow} \quad \begin{array}{c}\mathcal{E}'\\ \Delta, \Delta', \Delta'' \Vdash C\end{array}$$

Note that this is somewhat informal; a more formal account would label linear hypotheses with distinct variables, introduce proof terms, and carry out the corresponding substitution on the proof terms. We omit this here due to space considerations since it is rather standard. We also omitted the label *true*; we will often do so in the remainder of this paper when it is clear from context.

Conversely, local completeness shows that the elimination rules are not too weak—we can eliminate a propositional connective and reconstitute the judgment from the resulting evidence. We show this by means of a local expansion, transforming an arbitrary proof of a proposition to one that explicitly introduces its main connective.

$$\begin{array}{c}\mathcal{D}\\ \Delta \Vdash A \otimes B\end{array} \quad \underset{E}{\Longrightarrow} \quad \frac{\begin{array}{c}\mathcal{D}\\ \Delta \Vdash A \otimes B\end{array} \quad \dfrac{\overline{A \Vdash A} \quad \overline{B \Vdash B}}{A, B \Vdash A \otimes B} \otimes I}{\Delta \Vdash A \otimes B} \otimes E$$

Because locally sound rules for a connective do not create spurious evidence, we can characterize the connectives *independently* of each other, relying only on judgmental concepts and their introduction and elimination rules. The computational interpretation (β-reduction) of proofs depends on local reduction, and local expansion provides a source of canonical forms in logical frameworks (η-expansion).

The remaining multiplicative ($\mathbf{1}$, $\multimap$) and additive ($\&$, $\top$, $\oplus$, $\mathbf{0}$) connectives of intuitionistic linear logic have justifications similar to that of $\otimes$. We skip the individual rules and just note that we need no further judgmental constructs or principles for this fragment of linear logic.

2.2 Validity

The logic of the previous section permits only linear hypotheses. Therefore, it is too weak to embed ordinary intuitionistic or classical logic. To allow such embeddings, we introduce a new judgment, "*A is valid*", which we write A *valid*. We view validity not as a primitive judgment, but as a *categorical judgment* derived from truth in the absence of linear hypotheses. Similar notions of categorical judgment apply in a wide variety of logics [11, 23].

If $\cdot \Vdash A$ *true, then* A *valid.*

In the resource interpretation, A *valid* means that we can achieve A *true* without consuming any resources. Dually, an assumption A *valid* allows generation of as many copies of A *true* as needed, including none at all. Stated structurally, hypotheses of the form A *valid* allow weakening and contraction. The resulting hypothetical judgment has some *unrestricted hypotheses*, familiar from ordinary logic, and some *linear hypotheses* as described in the preceding section. We write this as a judgment with two zones containing assumptions.

$$\underbrace{A_1 \ \mathit{valid}, \ldots, A_j \ \mathit{valid}}_{\Gamma} \mathbin{;} \underbrace{B_1 \ \mathit{true}, \ldots, B_k \ \mathit{true}}_{\Delta} \vdash C \ \mathit{true}$$

As has become standard practice, we use "$;$" to separate unrestricted hypotheses from linear hypotheses. Using our above definition of validity, we may write $\Gamma \vdash A \ \mathit{valid}$ to express $\Gamma ; \cdot \vdash A \ \mathit{true}$; however, because we define connectives via rules about their *truth* rather than *validity*, we avoid using $A \ \mathit{valid}$ as a conclusion of hypothetical judgments. Instead, we incorporate the categorical view of validity in its hypothesis rule.

$$\frac{}{\Gamma, A \ \mathit{valid} ; \cdot \vdash A \ \mathit{true}} \ \mathsf{hyp!}$$

We obtain a second substitution principle to account for unrestricted hypotheses: if we have a proof that A *valid* we can substitute it for assumptions A *valid*.

Principle 2 (Substitution).

1. Whenever $\Gamma ; \Delta \vdash A \ \mathit{true}$ and $\Gamma ; \Delta', A \ \mathit{true} \vdash C \ \mathit{true}$, also $\Gamma ; \Delta, \Delta' \vdash C \ \mathit{true}$.
2. Whenever $\Gamma ; \cdot \vdash A \ \mathit{true}$ and $\Gamma, A ; \Delta \vdash C \ \mathit{true}$, also $\Gamma ; \Delta \vdash C \ \mathit{true}$.

Note that while Δ and Δ' are joined, Γ remains the same, expressing the fact that unrestricted hypotheses may be used multiple times.

We internalize validity as the familiar modal operator $!A$. The introduction rule makes the modality clear by requiring *validity* of A in the premiss. For the elimination rule, if we have a proof of $!A$, then we allow A as an unrestricted hypothesis.

$$\frac{\Gamma ; \cdot \vdash A}{\Gamma ; \cdot \vdash !A} \ !I \qquad \frac{\Gamma ; \Delta \vdash !A \quad \Gamma, A ; \Delta' \vdash C}{\Gamma ; \Delta, \Delta' \vdash C} \ !E$$

To establish local soundness, we use the new substitution principle to substitute $\mathcal{D}$ for uses of A in $\mathcal{E}$ to obtain $\mathcal{E}'$. For local completeness, we require the new hypothesis rule.

$$\frac{\dfrac{\overset{\mathcal{D}}{\Gamma ; \cdot \vdash A}}{\Gamma ; \cdot \vdash !A} \ !I \qquad \overset{\mathcal{E}}{\Gamma, A ; \Delta \vdash C}}{\Gamma ; \Delta \vdash C} \ !E \quad \overset{R}{\Longrightarrow} \quad \overset{\mathcal{E}'}{\Gamma ; \Delta \vdash C}$$

$$\overset{\mathcal{D}}{\Gamma ; \Delta \vdash !A} \quad \underset{E}{\Longrightarrow} \quad \frac{\overset{\mathcal{D}}{\Gamma ; \Delta \vdash !A} \qquad \dfrac{\dfrac{}{\Gamma, A ; \cdot \vdash A} \ \mathsf{hyp!}}{\Gamma, A ; \cdot \vdash !A} \ !I}{\Gamma ; \Delta \vdash !A} \ !E$$

The modal operator ! introduces a slight redundancy in the logic, in that we can show $\mathbf{1} \equiv !\top$. We leave $\mathbf{1}$ as a primitive in JILL because it has a definition in terms of introduction and elimination rules in the purely multiplicative fragment, that is, without additive connectives or validity.

2.3 Possibility

The system so far considers only conclusions of the form $A\ \mathit{true}$, which does not suffice to express a contradiction among the hypotheses and therefore negation. In the usual view, contradictory hypotheses describe a condition where an actual proof of the conclusion becomes unnecessary. However, such a view violates linearity of the conclusion. Another approach would be to define $\neg A = A \multimap \mathbf{0}$, as in Girard's translation from intuitionistic logic to classical linear logic, but then we give up all pretense of linearity because $A\ \mathit{true}$ and $\neg A\ \mathit{true}$ prove anything, independently of any other linear hypotheses that may remain.

To develop the notion of *linear contradiction*, we introduce a new judgment of *possibility*. Intuitively, a proof of $A\ \mathit{poss}$ defines *either* a proof of $A\ \mathit{true}$ *or* exhibits a linear contradiction among the hypotheses. We allow conclusions of the form $A\ \mathit{poss}$ in hypothetical judgments but eliminate it from consideration among the hypotheses. Dual to validity, we characterize $A\ \mathit{poss}$ via its substitution principle.

Principle 3 (Substitution for Possibility).
Whenever $\Gamma \mathrel{;} \Delta \vdash A\ \mathit{poss}$ and $\Gamma \mathrel{;} A\ \mathit{true} \vdash C\ \mathit{poss}$, also $\Gamma \mathrel{;} \Delta \vdash C\ \mathit{poss}$.

We justify this principle as follows. Assume $A\ \mathit{poss}$. Then, there may exist a contradiction among the assumptions, in which case also $C\ \mathit{poss}$. On the other hand, we may actually have $A\ \mathit{true}$, in which case any judgment $C\ \mathit{poss}$ we can derive from it is evident.

Two tempting generalizations of this substitution principle turn out to conflict with linearity. If for the second assumption we admit $\Gamma \mathrel{;} \Delta', A\ \mathit{true} \vdash C\ \mathit{poss}$, with the conclusion $\Gamma \mathrel{;} \Delta, \Delta' \vdash C\ \mathit{poss}$, then we weaken incorrectly if in fact we did not know that $A\ \mathit{true}$, but had contradictory hypotheses. The other incorrect generalization would be to replace $C\ \mathit{poss}$ with $C\ \mathit{true}$. This is also unsound if the assumptions were contradictory, because we would not have any evidence for $C\ \mathit{true}$.

Our explanation of possibility requires a new rule to conclude $A\ \mathit{poss}$ if $A\ \mathit{true}$.

$$\frac{\Gamma \mathrel{;} \Delta \vdash A\ \mathit{true}}{\Gamma \mathrel{;} \Delta \vdash A\ \mathit{poss}}\ \mathsf{poss}$$

The situation here is dual to validity. With validity, we added a new hypothesis rule but no explicit rule to conclude $A\ \mathit{valid}$, because we consider validity only as an assumption. With possibility, we added an explicit rule to conclude $A\ \mathit{poss}$ but no new hypothesis rule, because we consider possibility only as a conclusion.

The previous substitution principles for truth and validity (2) require an update for possibility judgments on the right. We consolidate all cases of the substitution principle below, using J schematically to stand for either $C\ \mathit{true}$ or $C\ \mathit{poss}$.

Principle 4 (Substitution).

1. Whenever $\Gamma \mathrel{;} \Delta \vdash A\ \mathit{poss}$ and $\Gamma \mathrel{;} A\ \mathit{true} \vdash C\ \mathit{poss}$, also $\Gamma \mathrel{;} \Delta \vdash C\ \mathit{poss}$.
2. Whenever $\Gamma \mathrel{;} \Delta \vdash A\ \mathit{true}$ and $\Gamma \mathrel{;} \Delta', A\ \mathit{true} \vdash J$, also $\Gamma \mathrel{;} \Delta, \Delta' \vdash J$.
3. Whenever $\Gamma \mathrel{;} \cdot \vdash A\ \mathit{true}$ and $\Gamma, A\ \mathit{valid} \mathrel{;} \Delta \vdash J$, also $\Gamma \mathrel{;} \Delta \vdash J$.

Similar to our treatment of validity, we internalize possibility as a proposition $?A$.

$$\frac{\Gamma \mathrel{;} \Delta \vdash A\ \mathit{poss}}{\Gamma \mathrel{;} \Delta \vdash ?A\ \mathit{true}}\ ?I \qquad \frac{\Gamma \mathrel{;} \Delta \vdash ?A\ \mathit{true} \quad \Gamma \mathrel{;} A\ \mathit{true} \vdash C\ \mathit{poss}}{\Gamma \mathrel{;} \Delta \vdash C\ \mathit{poss}}\ ?E$$

Judgmental Rules

$$\frac{}{\Gamma \mathbin{;} A \vdash A}\ \mathsf{hyp} \qquad \frac{}{\Gamma, A \mathbin{;} \cdot \vdash A}\ \mathsf{hyp!} \qquad \frac{\Gamma \mathbin{;} \Delta \vdash A}{\Gamma \mathbin{;} \Delta \vdash A\ \mathit{poss}}\ \mathsf{poss}$$

Multiplicative Connectives

$$\frac{\Gamma \mathbin{;} \Delta \vdash A \quad \Gamma \mathbin{;} \Delta' \vdash B}{\Gamma \mathbin{;} \Delta, \Delta' \vdash A \otimes B}\ \otimes I \qquad \frac{\Gamma \mathbin{;} \Delta \vdash A \otimes B \quad \Gamma \mathbin{;} \Delta', A, B \vdash J}{\Gamma \mathbin{;} \Delta, \Delta' \vdash J}\ \otimes E$$

$$\frac{}{\Gamma \mathbin{;} \cdot \vdash \mathbf{1}}\ \mathbf{1}I \qquad \frac{\Gamma \mathbin{;} \Delta \vdash \mathbf{1} \quad \Gamma \mathbin{;} \Delta' \vdash J}{\Gamma \mathbin{;} \Delta, \Delta' \vdash J}\ \mathbf{1}E$$

$$\frac{\Gamma \mathbin{;} \Delta, A \vdash B}{\Gamma \mathbin{;} \Delta \vdash A \multimap B}\ \multimap I \qquad \frac{\Gamma \mathbin{;} \Delta \vdash A \multimap B \quad \Gamma \mathbin{;} \Delta' \vdash A}{\Gamma \mathbin{;} \Delta, \Delta' \vdash B}\ \multimap E$$

Additive Connectives

$$\frac{\Gamma \mathbin{;} \Delta \vdash A \quad \Gamma \mathbin{;} \Delta \vdash B}{\Gamma \mathbin{;} \Delta \vdash A \mathbin{\&} B}\ \&I \qquad \frac{\Gamma \mathbin{;} \Delta \vdash A \mathbin{\&} B}{\Gamma \mathbin{;} \Delta \vdash A}\ \&E_1 \qquad \frac{\Gamma \mathbin{;} \Delta \vdash A \mathbin{\&} B}{\Gamma \mathbin{;} \Delta \vdash B}\ \&E_2$$

$$\frac{}{\Gamma \mathbin{;} \Delta \vdash \top}\ \top I \qquad \frac{\Gamma \mathbin{;} \Delta \vdash \mathbf{0}}{\Gamma \mathbin{;} \Delta, \Delta' \vdash J}\ \mathbf{0}E$$

$$\frac{\Gamma \mathbin{;} \Delta \vdash A}{\Gamma \mathbin{;} \Delta \vdash A \oplus B}\ \oplus I_1 \qquad \frac{\Gamma \mathbin{;} \Delta \vdash B}{\Gamma \mathbin{;} \Delta \vdash A \oplus B}\ \oplus I_2 \qquad \frac{\Gamma \mathbin{;} \Delta \vdash A \oplus B \quad \Gamma \mathbin{;} \Delta', A \vdash J \quad \Gamma \mathbin{;} \Delta', B \vdash J}{\Gamma \mathbin{;} \Delta, \Delta' \vdash J}\ \oplus E$$

Exponentials

$$\frac{\Gamma \mathbin{;} \cdot \vdash A}{\Gamma \mathbin{;} \cdot \vdash {!}A}\ {!}I \qquad \frac{\Gamma \mathbin{;} \Delta \vdash {!}A \quad \Gamma, A \mathbin{;} \Delta' \vdash J}{\Gamma \mathbin{;} \Delta, \Delta' \vdash J}\ {!}E$$

$$\frac{\Gamma \mathbin{;} \Delta \vdash A\ \mathit{poss}}{\Gamma \mathbin{;} \Delta \vdash {?}A}\ {?}I \qquad \frac{\Gamma \mathbin{;} \Delta \vdash {?}A \quad \Gamma \mathbin{;} A \vdash C\ \mathit{poss}}{\Gamma \mathbin{;} \Delta \vdash C\ \mathit{poss}}\ {?}E$$

Fig. 1. Natural deduction for JILL

Local reduction and expansion for ? demonstrate the new case of the substitution principle (4.1) applied to $\mathcal{D}$ and $\mathcal{E}$ to obtain $\mathcal{E}'$.

$$\frac{\dfrac{\overset{\mathcal{D}}{\Gamma \mathbin{;} \Delta \vdash A\ \mathit{poss}}}{\Gamma \mathbin{;} \Delta \vdash {?}A\ \mathit{true}}\ {?}I \qquad \overset{\mathcal{E}}{\Gamma \mathbin{;} A\ \mathit{true} \vdash C\ \mathit{poss}}}{\Gamma \mathbin{;} \Delta \vdash C\ \mathit{poss}}\ {?}E \quad \overset{R}{\Longrightarrow} \quad \overset{\mathcal{E}'}{\Gamma \mathbin{;} \Delta \vdash C\ \mathit{poss}}$$

$$\overset{\mathcal{D}}{\Gamma \mathbin{;} \Delta \vdash {?}A\ \mathit{true}} \quad \underset{E}{\Longrightarrow} \quad \dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma \mathbin{;} \Delta \vdash {?}A\ \mathit{true}} \qquad \dfrac{\dfrac{}{\Gamma \mathbin{;} A\ \mathit{true} \vdash A\ \mathit{true}}\ \mathsf{hyp}}{\Gamma \mathbin{;} A\ \mathit{true} \vdash A\ \mathit{poss}}\ \mathsf{poss}}{\Gamma \mathbin{;} \Delta \vdash A\ \mathit{poss}}\ {?}E}{\Gamma \mathbin{;} \Delta \vdash {?}A\ \mathit{true}}\ {?}I$$

The system so far with the primitive connectives $\otimes$, $\mathbf{1}$, $\&$, $\top$, $\oplus$, $\mathbf{0}$, $!$, and $?$ we call JILL for *judgmental intuitionistic linear logic*. Figure 1 lists the complete set of rules for these connectives. As a minor complication, various rules must now admit either a judgment $C\ \mathit{true}$ or $C\ \mathit{poss}$. For example, the rule for $\otimes$ elimination takes the schematic form

$$\frac{\Gamma \mathbin{;} \Delta \vdash A \otimes B \quad \Gamma \mathbin{;} \Delta', A, B \vdash J}{\Gamma \mathbin{;} \Delta, \Delta' \vdash J} \otimes E$$

where we intend J as either C *true* or C *poss*. The substitution principle already uses such a schematic presentation, so all local reductions remain correct.

2.4 Negation and Contradiction

Conceptually, in order to prove $\neg A$ *true*, we would like to assume A *true* and derive a contradiction. As remarked before, we want this contradiction to consume resources linearly, in order to distinguish it from the (non-linear) negation of A, definable as $A \multimap \mathbf{0}$. In other words, this contradiction should correspond to multiplicative falsehood, not additive falsehood ($\mathbf{0}$). This requirement suggests a conservative extension of JILL with the following additional judgment:

$$\Gamma \mathbin{;} \Delta \vdash \cdot$$

with the meaning that the hypotheses are (linearly) contradictory. Because we have described contradictory hypotheses as one of the possibilities for the judgment C *poss*, we obtain a *right-weakening* rule:

$$\frac{\Gamma \mathbin{;} \Delta \vdash \cdot}{\Gamma \mathbin{;} \Delta \vdash C \ \mathit{poss}} \ \mathsf{poss}'$$

The pair of rules poss and poss′ completely characterize our interpretation of C *poss* as either C *true* or a condition of contradictory hypotheses. The right-hand side J in various elimination rules (Fig. 1) and substitution principles (4) must extend to allow for this new judgment. In particular, the case for possibility (4.1) must also allow for $\cdot$ in addition to C *poss*.

Multiplicative contradiction $\bot$ internalizes this new hypothetical judgment, having obvious introduction and elimination rules.

$$\frac{\Gamma \mathbin{;} \Delta \vdash \cdot}{\Gamma \mathbin{;} \Delta \vdash \bot \ \mathit{true}} \ \bot I \qquad \frac{\Gamma \mathbin{;} \Delta \vdash \bot \ \mathit{true}}{\Gamma \mathbin{;} \Delta \vdash \cdot} \ \bot E$$

Multiplicative negation $\neg$ is also straightforward, having a multiplicative elimination rule.

$$\frac{\Gamma \mathbin{;} \Delta, A \vdash \cdot}{\Gamma \mathbin{;} \Delta \vdash \neg A \ \mathit{true}} \ \neg I \qquad \frac{\Gamma \mathbin{;} \Delta \vdash A \ \mathit{true} \quad \Gamma \mathbin{;} \Delta' \vdash \neg A \ \mathit{true}}{\Gamma \mathbin{;} \Delta, \Delta' \vdash \cdot} \ \neg E$$

Note that it is possible to view poss′ as an admissible structural rule of weakening on the right by allowing the conclusion of $\bot E$ and $\neg E$ to be either $\cdot$ or C *poss*. We take this approach in the next section.

Local soundness and completeness of the rules for $\bot$ and $\neg$ are easily verified, so we omit them here. Instead, we note that in JILL, we can define $\bot$ and $\neg$ notationally (as propositions).

$$\bot \stackrel{\text{def}}{=} \mathbf{?0} \qquad \neg A \stackrel{\text{def}}{=} A \multimap \bot$$

Thus, we have no need to add empty conclusions to JILL in order to define linear contradiction and negation. Note, however, that in the absence of additive contradiction $\mathbf{0}$ this definition is impossible. However, the extension with empty conclusions does not require additive connectives, so negation can be seen as belonging to the *multiplicative-exponential* fragment of JILL.

Without giving the straightforward verification of soundness and completeness of the definitions for $\bot$ and $\neg$ with respect to the extended judgment $\Gamma ; \Delta \vdash \cdot$, we present the following proof of $A \otimes \neg A \multimap ?C$ for arbitrary C.

$$
\dfrac{\dfrac{}{\cdot ; A \otimes \neg A \vdash A \otimes \neg A}\,\mathsf{hyp} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\cdot ; \neg A \vdash \neg A}\,\mathsf{hyp} \quad \dfrac{}{\cdot ; A \vdash A}\,\mathsf{hyp}}{\cdot ; A, \neg A \vdash ?\mathbf{0}}\,{\multimap}E \quad \dfrac{\dfrac{}{\cdot ; \mathbf{0} \vdash \mathbf{0}}\,\mathsf{hyp}}{\cdot ; \mathbf{0} \vdash C \;\; poss}\,\mathbf{0}E}{\cdot ; A, \neg A \vdash C \;\; poss}\,?E}{\cdot ; A, \neg A \vdash ?C}\,?I}{\cdot ; \cdot \vdash A \otimes \neg A \multimap ?C}\,{\otimes}E
$$

If in this derivation we try to replace $?C$ by C, then the instance of $?E$ becomes inapplicable, for this rule requires possibility on the right. To show formally that indeed $A \otimes \neg A \multimap C$ cannot hold for arbitrary C (*i.e.* $\neg$ behaves *multiplicatively*), we rely on the sequent calculus for JILL in Sec. 3.

2.5 A Modal Disjunction

Because natural deduction *a priori* admits only a single conclusion, a purely multiplicative disjunction seems conceptually difficult. Generalizing the right-hand side to admit more than one *true* proposition would violate either linearity or the intuitionistic interpretation in our setting. However, multiple conclusions do not necessarily conflict with natural deduction (see, for example, [20]), even for intuitionistic [26] and linear logics [16]. Indeed, we can readily incorporate such an approach in our judgmental framework by introducing a new judgment form, $C_1 \;\; poss \mid \cdots \mid C_k \;\; poss$, on the right-hand side of a hypothetical judgment. This judgment means that *either* the hypotheses are contradictory *or* one of the C_i is true. This gives the following rule for possibility (replacing the earlier poss)

$$
\dfrac{\Gamma ; \Delta \vdash C \;\; true}{\Gamma ; \Delta \vdash C \;\; poss \mid \Sigma}\,\mathsf{poss}
$$

where Σ stands for some alternation $C_1 \;\; poss \mid \cdots \mid C_k \;\; poss$, with free exchange assumed for "$|$". Since introduction rules define truth, this rule forces a commitment to the truth of a particular proposition, which retains both the intuitionistic and linear character of the logic. Structurally, Σ behaves as the dual of Γ on the right-hand side of hypothetical judgments, with weakening and contradiction as admissible structural rules. The multiplicative contradiction sketched in the previous section becomes a special case with *no* possible conclusions, *i.e.*, an empty Σ. We obtain some unsurprising substitution principles that we omit here.

Armed with this new judgment form, we define a multiplicative and modal disjunction $\parr$ with the following rules:

$$
\dfrac{\Gamma ; \Delta \vdash A \;\; poss \mid B \;\; poss}{\Gamma ; \Delta \vdash A \parr B \;\; true}\,{\parr}I \qquad \dfrac{\Gamma ; \Delta \vdash A \parr B \;\; true \quad \Gamma ; A \;\; true \vdash \Sigma \quad \Gamma ; B \;\; true \vdash \Sigma}{\Gamma ; \Delta \vdash \Sigma}\,{\parr}E
$$

By the nature of poss and the substitution principles, this disjunction has both linear and modal aspects.

Instead of completing the details of this sketch, we proceed in a manner similar to the previous section. Just as with multiplicative falsehood, it turns out that we can define $\parr$ notationally in JILL by combining additive disjunction with the ?-modality.

$$
A \parr B \quad \stackrel{\text{def}}{=} \quad ?(A \oplus B)
$$

This definition allows us to retain our presentation of JILL (Fig. 1) with a single conclusion. On the other hand, this definition requires the additive disjunction $\oplus$, so the definitional view does not hold for the multiplicative-exponential fragment of JILL.

In either definition, judgmental or notational, $⅋$ does not correspond exactly to the multiplicative disjunction from classical linear logic (CLL) or FILL because of its modal nature. For example, $\bot$ does not function as the unit of $⅋$, but we instead have the equivalence $A ⅋ \bot \dashv\vdash ?A$. Other laws, such as associativity or commutativity of $⅋$, however, do hold as expected.

2.6 Other Connectives and Properties

To summarize the preceding two sections, $\bot$ and $⅋$ have two different presentations. One requires a generalization of the judgment forms and presents them via their introduction and elimination rules. The other uses $?$, $\oplus$, and $\mathbf{0}$ to define them notationally. The fact that both explanations are viable and coincide confirms their status as logical connectives, not just abbreviations.

Like with intuitionistic logic, we can extend JILL with other connectives, either via introduction and elimination rules or directly via notational definitions. Two new forms of implication corresponding to the two modals ! and ? appear particularly useful.

name	**proposition**	**definition**
unrestricted implication	$A \supset B$	$!A \multimap B$
partial implication	$A \rightsquigarrow B$	$A \multimap ?B$

Under the Curry-Howard isomorphism, $A \supset B$ corresponds to the type of function that uses its argument arbitrarily often or possibly never. Similarly, $A \rightsquigarrow B$ corresponds to a *linear partial* function from type A to B. Other types such as $!A \multimap ?B$ can be given a sensible interpretation, in this case simply the partial functions from A to B. We expect the ? modality and various partial function types to be particularly useful when recursion and perhaps also effectful computations are added to obtain more practical languages, yet one does not want to sacrifice linearity entirely.

We close this section by stating explicitly the structural properties and the validity of the substitution principle.

Theorem 1 (Structural Properties). *JILL satisfies the substitution principles 4. Additionally, the following structural properties hold.*

1. *If $\Gamma \mathbin{;} \Delta \vdash C$ true then Γ, A valid $\mathbin{;} \Delta \vdash C$ true.* *(weakening)*
2. *If Γ, A valid, A valid $\mathbin{;} \Delta \vdash C$ true then Γ, A valid $\mathbin{;} \Delta \vdash C$ true.* *(contraction)*

Proof. By straightforward structural inductions. The substitution principle for possibility

$\Gamma \mathbin{;} \Delta \vdash A$ poss and $\Gamma \mathbin{;} A$ true $\vdash C$ poss, also $\Gamma \mathbin{;} \Delta \vdash C$ poss.

requires a somewhat unusual proof in that we have an induction over the structure of both given derivations and not just the second. This is not unexpected, however, given the analogy to the judgmental system for modal logic [23]. □

3 Sequent Calculus for JILL

Critical to the understanding of logical connectives is that the meaning of a proposition depends only on its constituents. Martin-Löf states [19, Page 27] that "*the meaning of a proposition is determined by [...] what counts as a verification of it*", but he does not elaborate on the notion of a verification. It seems clear that as a minimal condition, verifications must refer only to the propositions constituting the judgment they establish. In other words, they must obey the subformula property. We argue for a stronger condition that comes directly from the justification of rules, where we do not refer to any extraneous features of any particular proof.

Every logical inference in a verification must proceed purely by decomposition of one logical connective.

For the natural deduction view, introduction rules should only decompose the goal we are trying to achieve, while elimination rules should only decompose the assumptions we have. Since introduction rules have the subformula property when read bottom-up, and elimination rules have the subformula property when read top-down, any proof in this style will therefore satisfy this condition. The notion of a verification can thus be formalized directly on natural deductions (see, for example, [27]), but we take a different approach here and formalize verifications as *cut-free proofs in the sequent calculus*. Not only is it immediately evident that the sequent calculus satisfies our condition, but it is easier to prove the correctness of the interpretations of classical linear logic in Sec. 4, which is usually presented in the form of sequents.

The fundamental transformation giving rise to the sequent calculus splits the judgment A *true* into a pair of judgments for resources and goals, A *res* and A *goal*. We then consider linear hypothetical judgments of the form

$$B_1 \ \mathit{res}, \ldots, B_k \ \mathit{res} \Vdash C \ \mathit{goal}.$$

We never consider C *goal* as a hypothesis or B *res* as a conclusion. Therefore, we do not have a hypothesis rule with the same judgment on both sides of $\Vdash$, like for natural deduction. Rather, if we have the resource A we can achieve goal A; *i.e.*, we have a rule relating the judgments A *res* and A *goal*.

$$\frac{}{A \ \mathit{res} \Vdash A \ \mathit{goal}} \ \mathsf{init}$$

Because we do not allow resources on the right and goals on the left, we cannot write its dual as A *goal* $\Vdash A$ *res*. Instead, we obtain a form of cut as a proper dual of the init rule.

Principle 5 (Cut).
Whenever $\Delta \Vdash A \ \mathit{goal}$ and $\Delta', A \ \mathit{res} \Vdash C \ \mathit{goal}$, also $\Delta, \Delta' \Vdash C \ \mathit{goal}$.

In words, if we have achieved a goal A, then we may justifiably use A as a resource. Because we distinguish resources and goals, cut does not correspond exactly to a substitution principle in natural deduction. But, similar to the substitution principles, cut must remain an admissible rule in order to preserve our view of verification; otherwise, a proof can refer to an arbitrary cut formula A that does not occur in the conclusion. Similarly, if we did not distinguish two judgments, the interpretation of hypothetical judgments would force the collapse back to natural deduction.

To capture all of JILL as a sequent calculus, we also need to account for validity and possibility. Fortunately, we already restrict their occurrence in hypothetical judgments—to the left for validity and to the right for possibility. These judgments therefore do not require splits. We obtain the following general judgment forms, which we call *sequents* following Gentzen's terminology.

$$A_1 \ \mathit{valid}, \ldots, A_j \ \mathit{valid} \,;\, B_1 \ \mathit{res}, \ldots, B_k \ \mathit{res} \Longrightarrow C \ \mathit{goal}$$
$$A_1 \ \mathit{valid}, \ldots, A_j \ \mathit{valid} \,;\, B_1 \ \mathit{res}, \ldots, B_k \ \mathit{res} \Longrightarrow C \ \mathit{poss}$$

As before, we write the left-hand side schematically as $\Gamma \,;\, \Delta$. The division of the left-hand side into zones allows us to leave the judgment labels implicit. We employ a symmetric device on the right-hand side, representing C *goal* by $C \,;\, \cdot$ and C *poss* by $\cdot \,;\, C$. For left rules where the actual form of the right-hand side often does not matter, we write it schematically as γ. We use $\Longrightarrow$ instead of $\vdash$ for the hypothetical judgment to visually distinguish the sequent calculus from natural deduction.

We now systematically construct the rules defining the judgments of the sequent calculus. The introduction rules from natural deduction turn into *right rules* that operate only on goals and retain their bottom-up interpretation. For the elimination rules, we reverse the direction and have them operate only on resources, thus turning them into *left rules*, also read bottom-up. Rules in the sequent calculus therefore have a uniform bottom-up interpretation, unlike the rules for natural deduction.

With the inclusion of *valid* and *poss* judgments, the init rule has the following most general form.

$$\frac{}{\Gamma \,⨾\, A \Longrightarrow A \,⨾\, \cdot}\ \mathsf{init}$$

We allow copying valid hypotheses in Γ into the linear context Δ (reading bottom-up) by means of a copy rule. This rule corresponds to hyp! of natural deduction.

$$\frac{\Gamma, A \,⨾\, \Delta, A \Longrightarrow \gamma}{\Gamma, A \,⨾\, \Delta \Longrightarrow \gamma}\ \mathsf{copy}$$

Finally, we include the counterpart of the poss, which in the sequent calculus promotes a possibility goal $\cdot \,⨾\, C$ in the conclusion into a true goal $C \,⨾\, \cdot$ in the premiss.

$$\frac{\Gamma \,⨾\, \Delta \Longrightarrow C \,⨾\, \cdot}{\Gamma \,⨾\, \Delta \Longrightarrow \cdot \,⨾\, C}\ \mathsf{promote}$$

Figure 2 lists the remaining left and right rules for the various connectives. Structurally, weakening and contraction of the valid context Γ continue to hold; we omit the easy verification. Cut comes in three forms, dualizing init, copy, and promote, respectively.

Principle 6 (Cut).

1. Whenever $\Gamma \,⨾\, \cdot \Longrightarrow A \,⨾\, \cdot$ and $\Gamma, A \,⨾\, \Delta \Longrightarrow \gamma$, also $\Gamma \,⨾\, \Delta \Longrightarrow \gamma$.
2. Whenever $\Gamma \,⨾\, \Delta \Longrightarrow A \,⨾\, \cdot$ and $\Gamma \,⨾\, \Delta', A \Longrightarrow \gamma$, also $\Gamma \,⨾\, \Delta, \Delta' \Longrightarrow \gamma$.
3. Whenever $\Gamma \,⨾\, \Delta \Longrightarrow \cdot \,⨾\, A$ and $\Gamma \,⨾\, A \Longrightarrow \cdot \,⨾\, C$, also $\Gamma \,⨾\, \Delta \Longrightarrow \cdot \,⨾\, C$.

With our logic fixed, we *prove* the admissibility of the cut principles.

Theorem 2 (Admissibility of Cut).
The principles of cut (Prin. 6) are admissible rules in JILL.

Proof. First we name all the derivations (writing $\mathcal{D} :: J$ if $\mathcal{D}$ is a derivation of judgment J):

$\mathcal{D}_1 :: \Gamma \,⨾\, \cdot \Longrightarrow A \,⨾\, \cdot$	$\mathcal{E}_1 :: \Gamma, A \,⨾\, \Delta' \Longrightarrow \gamma$	$\mathcal{F}_1 :: \Gamma, \Delta' ⨾ \Longrightarrow \gamma$	(row 1)
$\mathcal{D}_2 :: \Gamma \,⨾\, \Delta \Longrightarrow A \,⨾\, \cdot$	$\mathcal{E}_2 :: \Gamma \,⨾\, \Delta', A \Longrightarrow \gamma$	$\mathcal{F}_2 :: \Gamma \,⨾\, \Delta, \Delta' \Longrightarrow \gamma$	(row 2)
$\mathcal{D}_3 :: \Gamma \,⨾\, \Delta \Longrightarrow \cdot \,⨾\, A$	$\mathcal{E}_3 :: \Gamma \,⨾\, A \Longrightarrow \cdot \,⨾\, C$	$\mathcal{F}_3 :: \Gamma \,⨾\, \Delta \Longrightarrow \cdot \,⨾\, C$	(row 3)

The computational content of the proof is a way to transform corresponding $\mathcal{D}$ and $\mathcal{E}$ into $\mathcal{F}$. For the purposes of the inductive argument, we may appeal to the induction hypotheses whenever

a. the cut formula A is strictly smaller;
b. the cut formula A remains the same, but we select the inductive hypothesis from row 3 for proofs in rows 2 and 1, or from row 2 in proofs of row 1;
c. the cut formula A and the derivations $\mathcal{E}$ remain the same, but the derivations $\mathcal{D}$ becomes smaller; or
d. the cut formula A and the derivations $\mathcal{D}$ remain the same, but the derivations $\mathcal{E}$ become smaller.

The cases in the inductive proof fall into the following classes, which we will explicitly name and for which we provide a characteristic case.

Judgmental Rules

$$\frac{}{\Gamma ; A \Longrightarrow A ; \cdot}\ \mathsf{init} \qquad \frac{\Gamma, A ; \Delta, A \Longrightarrow \gamma}{\Gamma, A ; \Delta \Longrightarrow \gamma}\ \mathsf{copy} \qquad \frac{\Gamma ; \Delta \Longrightarrow A ; \cdot}{\Gamma ; \Delta \Longrightarrow \cdot ; A}\ \mathsf{promote}$$

Multiplicative Connectives

$$\frac{\Gamma ; \Delta, A, B \Longrightarrow \gamma}{\Gamma ; \Delta, A \otimes B \Longrightarrow \gamma}\ \otimes L \qquad \frac{\Gamma ; \Delta \Longrightarrow A ; \cdot \quad \Gamma ; \Delta \Longrightarrow B ; \cdot}{\Gamma ; \Delta, \Delta' \Longrightarrow A \otimes B ; \cdot}\ \otimes R$$

$$\frac{\Gamma ; \Delta \Longrightarrow \gamma}{\Gamma ; \Delta, \mathbf{1} \Longrightarrow \gamma}\ \mathbf{1}L \qquad \frac{}{\Gamma ; \cdot \Longrightarrow \mathbf{1} ; \cdot}\ \mathbf{1}R$$

$$\frac{\Gamma ; \Delta \Longrightarrow A ; \cdot \quad \Gamma ; \Delta', B \Longrightarrow \gamma}{\Gamma ; \Delta, \Delta', A \multimap B \Longrightarrow \gamma}\ \multimap L \qquad \frac{\Gamma ; \Delta, A \Longrightarrow B ; \cdot}{\Gamma ; \Delta \Longrightarrow A \multimap B ; \cdot}\ \multimap R$$

Additive Connectives

$$\frac{\Gamma ; \Delta, A \Longrightarrow \gamma}{\Gamma ; \Delta, A \mathbin{\&} B \Longrightarrow \gamma}\ \&L_1 \qquad \frac{\Gamma ; \Delta, B \Longrightarrow \gamma}{\Gamma ; \Delta, A \mathbin{\&} B \Longrightarrow \gamma}\ \&L_2 \qquad \frac{\Gamma ; \Delta \Longrightarrow A ; \cdot \quad \Gamma ; \Delta \Longrightarrow B ; \cdot}{\Gamma ; \Delta \Longrightarrow A \mathbin{\&} B ; \cdot}\ \&R$$

$$\frac{}{\Gamma ; \Delta \Longrightarrow \top ; \cdot}\ \top R \qquad \frac{}{\Gamma ; \Delta, \mathbf{0} \Longrightarrow \gamma}\ \mathbf{0}L$$

$$\frac{\Gamma ; \Delta, A \Longrightarrow \gamma \quad \Gamma ; \Delta, B \Longrightarrow \gamma}{\Gamma ; \Delta, A \oplus B \Longrightarrow \gamma}\ \oplus L \qquad \frac{\Gamma ; \Delta \Longrightarrow A ; \cdot}{\Gamma ; \Delta \Longrightarrow A \mathbin{\&} B ; \cdot}\ \oplus R_1 \qquad \frac{\Gamma ; \Delta \Longrightarrow B ; \cdot}{\Gamma ; \Delta \Longrightarrow A \mathbin{\&} B ; \cdot}\ \oplus R_2$$

Exponentials

$$\frac{\Gamma, A ; \Delta \Longrightarrow \gamma}{\Gamma ; \Delta, !A \Longrightarrow \gamma}\ !L \qquad \frac{\Gamma ; \cdot \Longrightarrow A ; \cdot}{\Gamma ; \cdot \Longrightarrow !A ; \cdot}\ !R \qquad \frac{\Gamma ; A \Longrightarrow \cdot ; C}{\Gamma ; ?A \Longrightarrow \cdot ; C}\ ?L \qquad \frac{\Gamma ; \Delta \Longrightarrow \cdot ; A}{\Gamma ; \Delta \Longrightarrow ?A ; \cdot}\ ?R$$

Fig. 2. Sequent calculus for JILL

Initial Cuts. Here, we find an initial sequent in one of the two premisses, so we eliminate these cases directly. For example,

$$\mathcal{D}_2 = \frac{}{\Gamma ; A \Longrightarrow A ; \cdot}\ \mathsf{init} \qquad\qquad \mathcal{E}_2 \text{ arbitrary}$$

$\Delta = A$	this case
$\Gamma ; \Delta', A \Longrightarrow \gamma$	derivation $\mathcal{E}_2$

Principal Cuts. The cut formula A was just inferred by a right rule in the first premiss and a left rule in the second premiss. In these cases, we appeal to the induction hypotheses, possibly more than once on smaller cut formulas. For example,

$$\mathcal{D}_2 = \frac{\overset{\mathcal{D}_2'}{\Gamma ; \Delta \Longrightarrow \cdot ; A}}{\Gamma ; \Delta \Longrightarrow ?A ; \cdot}\ ?R \qquad\qquad \mathcal{E}_2 = \frac{\overset{\mathcal{E}_2'}{\Gamma ; A \Longrightarrow \cdot ; C}}{\Gamma ; ?A \Longrightarrow \cdot ; C}\ ?L$$

$\Gamma ; \Delta \Longrightarrow \cdot ; C$	by i.h. (row 3) (case a) on $\mathcal{D}_2'$ and $\mathcal{E}_2'$ using cut formula A

Copy Cut. We treat the cases for the cut! rule as right commutative cuts (below), except for the copy rule, where we require an appeal to an induction hypothesis on the same cut formula.

$$\mathcal{D}_1 \text{ arbitrary} \qquad\qquad \mathcal{E}_1 = \frac{\overset{\mathcal{E}_1'}{\Gamma, A ; \Delta', A \Longrightarrow \gamma}}{\Gamma, A ; \Delta' \Longrightarrow \gamma}\ \mathsf{copy}$$

$\Gamma ; \cdot \Longrightarrow A ; \cdot$ derivation $\mathcal{D}_1$
$\Gamma ; \Delta', A \Longrightarrow \gamma$ by i.h. (row 1) (case d) on A, $\mathcal{D}_1$, and $\mathcal{E}'_1$
$\Gamma ; \Delta' \Longrightarrow \gamma$ by i.h. (row 2) (case b) on A, $\mathcal{D}_1$, and above

Promote Cut. We treat the cases for the cut? rule as left commutative cuts (below), except for the promote rule, where we appeal to an inductive hypothesis with the same cut formula.

$$\mathcal{D}_3 = \cfrac{\begin{array}{c}\mathcal{D}'_3\\ \Gamma ; \Delta \Longrightarrow A ; \cdot\end{array}}{\Gamma ; \Delta \Longrightarrow \cdot ; A}\ \text{promote} \qquad\qquad \mathcal{E}_3 \text{ arbitrary}$$

$\Gamma ; A \Longrightarrow \gamma$ derivation $\mathcal{E}_3$
$\Gamma ; \Delta \Longrightarrow \gamma$ by i.h. (row 2) (case c) on A, $\mathcal{D}'_3$, and $\mathcal{E}_3$.

Left Commutative Cuts. The cut formula A exists as a side-formula of the last inference rule used in the derivation of the left premiss. In these cases, we appeal to the induction with the same cut formula but smaller left derivation. For example,

$$\mathcal{D}_2 = \cfrac{\begin{array}{c}\mathcal{D}'_2\\ \Gamma ; \Delta, B_1 \Longrightarrow A ; \cdot\end{array}}{\Gamma ; \Delta, B_1 \,\&\, B_2 \Longrightarrow A ; \cdot}\ \&L_1 \qquad\qquad \mathcal{E}_2 \text{ arbitrary}$$

$\Gamma ; \Delta', A \Longrightarrow \gamma$ derivation $\mathcal{E}_2$
$\Gamma ; \Delta, \Delta', B_1 \Longrightarrow \gamma$ by i.h. (row 2) (case c) on A, $\mathcal{D}'_2$, and $\mathcal{E}_2$
$\Gamma ; \Delta, \Delta', B_1 \,\&\, B_2 \Longrightarrow \gamma$ by $\&L_1$

Right Commutative Cuts. The cut formula A exists as a side-formula of the last inference rule used in the derivation of the right premiss. In these cases, we appeal to the induction hypotheses with the same cut formula but smaller right derivation. For example,

$$\mathcal{D}_2 \text{ arbitrary} \qquad\qquad \mathcal{E}_2 = \cfrac{\begin{array}{c}\mathcal{E}'_2\\ \Gamma ; \Delta', A \Longrightarrow C_1 ; \cdot\end{array}}{\Gamma ; \Delta', A \Longrightarrow C_1 \oplus C_2 ; \cdot}\ \oplus R_1$$

$\Gamma ; \Delta \Longrightarrow A ; \cdot$ derivation $\mathcal{D}_2$
$\Gamma ; \Delta, \Delta' \Longrightarrow C_1 ; \cdot$ by i.h. (row 2) (case d) on A, $\mathcal{D}_2$, and $\mathcal{E}'_2$
$\Gamma ; \Delta, \Delta' \Longrightarrow C_1 \oplus C_2 ; \cdot$ by $\oplus R_1$

All cases in the induction belong to one of these categories. □

Comparing this proof of cut-admissibility for JILL with other proofs of cut-admissibility or cut-elimination in the literature, we find it remarkable that a nested structural induction suffices, requiring no additional restrictions or induction measures. Similar structural proofs for the admissibility of cut have been demonstrated for classical linear logic [21] and ordered logic [25].

Sequents have valid interpretations in the natural deduction calculus, which we state as a soundness theorem for the sequent calculus.

Theorem 3 (Soundness of $\Longrightarrow$ wrt $\vdash$).

1. *If $\Gamma ; \Delta \Longrightarrow C ; \cdot$, then $\Gamma ; \Delta \vdash C$ true.*
2. *If $\Gamma ; \Delta \Longrightarrow \cdot ; C$, then $\Gamma ; \Delta \vdash C$ poss.*

Proof. Proceed by simultaneous induction on the derivations $\mathcal{D}_1 :: \Gamma \,;\, \Delta \Longrightarrow C \,;\, \cdot$ and $\mathcal{D}_2 :: \Gamma \,;\, \Delta \Longrightarrow \cdot \,;\, C$. For the right rules, we appeal to the induction hypothesis and apply the corresponding introduction rule. For the left rules, we either directly construct a derivation or appeal to the substitution principle after applying the inductive hypothesis. We show in detail some representative cases:

1. The last rule in $\mathcal{D}_1$ is init, *i.e.*,

$$\mathcal{D}_1 = \frac{}{\Gamma \,;\, A \Longrightarrow A \,;\, \cdot}\ \mathsf{init}$$

$\Gamma \,;\, A \vdash A\ \ true$ by hyp

2. The last rule in $\mathcal{D}_1$ or $\mathcal{D}_2$ is copy, *i.e.*,

$$\mathcal{D}_1 \text{ or } \mathcal{D}_2 = \frac{\begin{array}{c}\mathcal{D}' \\ \Gamma, A \,;\, \Delta, A \Longrightarrow \gamma\end{array}}{\Gamma, A \,;\, \Delta \Longrightarrow \gamma}\ \mathsf{copy}$$

$\Gamma, A \,;\, \Delta, A \vdash \gamma$ by the i.h. on $\mathcal{D}'$
$\Gamma, A \,;\, \cdot \vdash A\ \ true$ by hyp!
$\Gamma, A \,;\, \Delta \vdash \gamma$ by the substitution principle for truth (4.2)

3. The last rule in $\mathcal{D}_2$ is promote, *i.e.*,

$$\mathcal{D}_2 = \frac{\begin{array}{c}\mathcal{D}_2' \\ \Gamma \,;\, \Delta \Longrightarrow C \,;\, \cdot\end{array}}{\Gamma \,;\, \Delta \Longrightarrow \cdot \,;\, C}\ \mathsf{promote}$$

$\Gamma \,;\, \Delta \vdash C\ \ true$ by the i.h. on $\mathcal{D}_2'$
$\Gamma \,;\, \Delta \vdash C\ \ poss$ by poss

4. The last rule in $\mathcal{D}_2$ is $?L$, *i.e.*,

$$\mathcal{D}_2 = \frac{\begin{array}{c}\mathcal{D}_2' \\ \Gamma \,;\, A \Longrightarrow \cdot \,;\, C\end{array}}{\Gamma \,;\, ?A \Longrightarrow \cdot \,;\, C}\ ?L$$

$\Gamma \,;\, A \vdash C\ \ poss$ by the i.h. on $\mathcal{D}_2'$
$\Gamma \,;\, ?A \vdash ?A\ \ true$ by hyp
$\Gamma \,;\, ?A \vdash C\ \ poss.$ by $?E$

5. The last rule in $\mathcal{D}_1$ or $\mathcal{D}_2$ is $\multimap L$, *i.e.*,

$$\mathcal{D}_1 \text{ or } \mathcal{D}_2 = \frac{\begin{array}{c}\mathcal{D}' \\ \Gamma \,;\, \Delta \Longrightarrow A \,;\, \cdot\end{array} \quad \begin{array}{c}\mathcal{D}'' \\ \Gamma \,;\, \Delta', B \Longrightarrow \gamma\end{array}}{\Gamma \,;\, \Delta, \Delta', A \multimap B \Longrightarrow \gamma}\ \multimap L$$

$\Gamma \,;\, \Delta \vdash A\ \ true$ by the i.h. on $\mathcal{D}'$
$\Gamma \,;\, A \multimap B \vdash A \multimap B\ \ true$ by hyp
$\Gamma \,;\, \Delta, A \multimap B \vdash B\ \ true$ by $\multimap E$
$\Gamma \,;\, \Delta', B \vdash \gamma$ by the i.h. on $\mathcal{D}''$
$\Gamma \,;\, \Delta, \Delta', A \multimap B \vdash \gamma$ by the substitution principle for truth (4.2)

6. The last rule in $\mathcal{D}_2$ is $?R$, *i.e.*,

$$\mathcal{D}_2 = \frac{\begin{array}{c}\mathcal{D}_2' \\ \Gamma \,;\, \Delta \Longrightarrow \cdot \,;\, A\end{array}}{\Gamma \,;\, \Delta \Longrightarrow ?A \,;\, \cdot}\ ?R$$

$$\Gamma \mathbin{;} \Delta \vdash A \;\; poss \qquad \text{by the i.h. on } \mathcal{D}_2'$$
$$\Gamma \mathbin{;} \Delta \vdash ?A \;\; true \qquad \text{by } ?I$$

All remaining cases have one of the above patterns. □

Note that from the judgmental point of view, this defines a *global completeness* property for natural deduction: every judgment that has a verification can indeed be proven.

Conversely, the cut-free sequent calculus is complete with respect to natural deduction. For this direction, we need the admissibility of cut.

Theorem 4 (Completeness of $\Longrightarrow$ wrt $\vdash$).

1. *If $\Gamma \mathbin{;} \Delta \vdash A$ true then $\Gamma \mathbin{;} \Delta \Longrightarrow A \mathbin{;} \cdot$.*
2. *If $\Gamma \mathbin{;} \Delta \vdash A$ poss, then $\Gamma \mathbin{;} \Delta \Longrightarrow \cdot \mathbin{;} A$.*

Proof. Proceed by simultaneous induction on the structure of the derivations $\mathcal{D}_1 :: \Gamma \mathbin{;} \Delta \vdash A \;\; true$ and $\mathcal{D}_2 :: \Gamma \mathbin{;} \Delta \vdash A \;\; poss$. The cases for the introduction rules (and poss) are mapped directly to the corresponding right rules (and promote, respectively). For the elimination rule, we have to appeal to cut-admissibility for truth (Thm. 2) to cut out the connective being eliminated. We show in detail some representative cases:

1. The last rule in $\mathcal{D}_1$ is hyp!, *i.e.*,

$$\mathcal{D}_1 = \frac{}{\Gamma, A \mathbin{;} \cdot \vdash A}\;\text{hyp!}$$

$$\Gamma, A \mathbin{;} A \Longrightarrow A \mathbin{;} \cdot \qquad \text{by init}$$
$$\Gamma, A \mathbin{;} \cdot \Longrightarrow A \mathbin{;} \cdot \qquad \text{by copy}$$

2. The last rule in $\mathcal{D}_1$ is hyp, *i.e.*,

$$\mathcal{D}_1 = \frac{}{\Gamma \mathbin{;} A \vdash A}\;\text{hyp}$$

$$\Gamma \mathbin{;} A \Longrightarrow A \mathbin{;} \cdot \qquad \text{by init}$$

3. The last rule in $\mathcal{D}_2$ is poss, *i.e.*,

$$\mathcal{D}_2 = \frac{\begin{array}{c}\mathcal{D}_2' \\ \Gamma \mathbin{;} \Delta \vdash A\end{array}}{\Gamma \mathbin{;} \Delta \vdash A \;\; poss}\;\text{poss}$$

$$\Gamma \mathbin{;} \Delta \Longrightarrow A \mathbin{;} \cdot \qquad \text{by i.h. on } \mathcal{D}_2'$$
$$\Gamma \mathbin{;} \Delta \Longrightarrow \cdot \mathbin{;} A \qquad \text{by promote}$$

4. The last rule in $\mathcal{D}_1$ is a multiplicative elimination rule, say $\multimap E$.

$$\mathcal{D}_1 = \frac{\begin{array}{c}\mathcal{D}_1' \\ \Gamma \mathbin{;} \Delta \vdash A \multimap B\end{array} \quad \begin{array}{c}\mathcal{D}_1'' \\ \Gamma \mathbin{;} \Delta' \vdash A\end{array}}{\Gamma \mathbin{;} \Delta, \Delta' \vdash B}\;\multimap E$$

$$\Gamma \mathbin{;} A \Longrightarrow A \mathbin{;} \cdot \qquad \text{by init}$$
$$\Gamma \mathbin{;} B \Longrightarrow B \mathbin{;} \cdot \qquad \text{by init}$$
$$\Gamma \mathbin{;} A, A \multimap B \Longrightarrow B \mathbin{;} \cdot \qquad \text{by } \multimap L$$
$$\Gamma \mathbin{;} \Delta \Longrightarrow A \multimap B \mathbin{;} \cdot \qquad \text{by i.h. on } \mathcal{D}_1'$$
$$\Gamma \mathbin{;} A, \Delta \Longrightarrow B \mathbin{;} \cdot \qquad \text{by cut (6.2) with cut formula } A \multimap B$$
$$\Gamma \mathbin{;} \Delta' \Longrightarrow A \mathbin{;} \cdot \qquad \text{by i.h. on } \mathcal{D}_1''$$
$$\Gamma \mathbin{;} \Delta', \Delta \Longrightarrow B \mathbin{;} \cdot \qquad \text{by cut (6.2) with cut formula } A$$

5. The last rule in $\mathcal{D}_1$ is an additive elimination rule, say $\& E_1$.

$$\mathcal{D}_1 = \dfrac{\begin{array}{c}\mathcal{D}_1' \\ \Gamma ⨾ \Delta \vdash A \mathbin{\&} B\end{array}}{\Gamma ⨾ \Delta \vdash A} \; \& E_1$$

$\Gamma ⨾ A \Longrightarrow A ⨾ \cdot$	by init
$\Gamma ⨾ A \mathbin{\&} B \Longrightarrow A ⨾ \cdot$	by $\& L_1$
$\Gamma ⨾ \Delta \Longrightarrow A \mathbin{\&} B ⨾ \cdot$	by i.h. on $\mathcal{D}_1'$
$\Gamma ⨾ \Delta \Longrightarrow A ⨾ \cdot$	by cut (6.2) with cut formula $A \mathbin{\&} B$

6. The last rule in $\mathcal{D}_1$ is $!E$, *i.e.*,

$$\mathcal{D}_1 = \dfrac{\begin{array}{c}\mathcal{D}_1' \\ \Gamma ⨾ \Delta \vdash {!A}\end{array} \quad \begin{array}{c}\mathcal{D}_1'' \\ \Gamma, A ⨾ \Delta' \vdash J\end{array}}{\Gamma ⨾ \Delta, \Delta' \vdash J} \; !E$$

Define $\ulcorner J \urcorner$ as

$$\ulcorner J \urcorner = \begin{cases} C ⨾ \cdot & \text{if } J \text{ is } C \ \mathit{true} \\ \cdot ⨾ C & \text{if } J \text{ is } C \ \mathit{poss} \end{cases}$$

Then,

$\Gamma, A ⨾ \Delta' \Longrightarrow \ulcorner J \urcorner$	by i.h. on $\mathcal{D}_1''$
$\Gamma ⨾ \Delta', !A \Longrightarrow \ulcorner J \urcorner$	by $!L$
$\Gamma ⨾ \Delta \Longrightarrow {!A} ⨾ \cdot$	by i.h. on $\mathcal{D}_1'$
$\Gamma ⨾ \Delta, \Delta' \Longrightarrow \ulcorner J \urcorner$	by cut (6.2) with cut formula $!A$

7. The last rule in $\mathcal{D}_2$ is $?E$, *i.e.*,

$$\mathcal{D}_2 = \dfrac{\begin{array}{c}\mathcal{D}_2' \\ \Gamma ⨾ \Delta \vdash {?A}\end{array} \quad \begin{array}{c}\mathcal{D}_2'' \\ \Gamma ⨾ A \vdash C \ \mathit{poss}\end{array}}{\Gamma ⨾ \Delta \vdash C \ \mathit{poss}} \; ?E$$

$\Gamma ⨾ A \Longrightarrow \cdot ⨾ C$	by i.h. on $\mathcal{D}_2''$
$\Gamma ⨾ {?A} \Longrightarrow \cdot ⨾ C$	by $?L$
$\Gamma ⨾ \Delta \Longrightarrow {?A} ⨾ \cdot$	by i.h. on $\mathcal{D}_2'$
$\Gamma ⨾ \Delta \Longrightarrow \cdot ⨾ C$	by cut (6.2) with cut formula $?A$.

All remaining cases have one of the above patterns. □

Note that this completeness theorem for cut-free sequent derivation proves a *global soundness* theorem for natural deduction: every judgment that has a natural deduction proof has a cut-free sequent derivation, that is, it has a *verification*.

It is also possible to split the judgment A *true* into A *intro* and A *elim*, roughly meaning that it has been established by an introduction or elimination rule, respectively. We can then define a normal form for natural deductions in which one can go from eliminations to introductions but not vice versa, which guarantees the subformula property. By adding commuting reductions to the local reductions, one can reduce every natural deduction to a normal form that can serve as a verification. On the whole we find this approach less perspicuous and more difficult to justify; thus our choice of the sequent calculus.

The cut-free sequent calculus is an easy source of theorems regarding JILL. For example, if we want to verify that $A \otimes \neg A \multimap C$ *true* cannot be proven for parameters A and C, we just explore all possibilities for a derivation of $\cdot ⨾ \cdot \Longrightarrow A \otimes \neg A \multimap C ⨾ \cdot$, each of which fails after only a few steps. The sequent calculus is also a useful point of departure for designing theorem proving procedures for intuitionistic linear logic.

4 Interpreting Classical Linear Logic

It is well known that intuitionistic logic is more expressive than classical logic because it makes finer distinctions and therefore has a richer set of connectives. This observation is usually formalized via a translation from classical logic to intuitionistic logic that preserves provability. A related argument has been made by Girard [13] regarding classical linear logic: it is more expressive than both intuitionistic and classical logic since we can easily interpret both of these. In this section we show that intuitionistic linear logic is yet more expressive than classical linear logic by giving a simple compositional translation that preserves the structure of proofs. Because there are fewer symmetries, we obtain a yet again richer set of connectives. For example, ? and ! cannot be defined in terms of each other via negation, echoing a related phenomenon in intuitionistic modal logic [23].

For our embedding, we use a one-sided sequent calculus for CLL with two zones as introduced by Andreoli [3], who proves it equivalent to Girard's formulation. This is closely related to our judgmental distinction between truth and validity. In the judgment $\xrightarrow{\text{CLL}} \Gamma \fatsemi \Delta$ (which may be read as $\xrightarrow{\text{CLL}} !\Gamma, \Delta$), we reuse the letters Γ and Δ, but note that they do *not* correspond directly to valid or true hypotheses. However, Γ satisfies weakening and contraction, while Δ does not. As usual, we take exchange for granted and rely on the admissibility of cut, which can be proven in a variety of ways, including a structural induction similar to the one for Thm. 2 (see also [3, 21]).

$$
\dfrac{}{\xrightarrow{\text{CLL}} \Gamma \fatsemi P, P^{\perp}}\ \text{init}
\qquad
\dfrac{\xrightarrow{\text{CLL}} \Gamma, A \fatsemi \Delta, A}{\xrightarrow{\text{CLL}} \Gamma, A \fatsemi \Delta}\ \text{copy}
$$

$$
\dfrac{\xrightarrow{\text{CLL}} \Gamma \fatsemi \Delta_1, A \quad \xrightarrow{\text{CLL}} \Gamma \fatsemi \Delta_2, B}{\xrightarrow{\text{CLL}} \Gamma \fatsemi \Delta_1, \Delta_2, A \otimes B}\ \otimes
\qquad
\dfrac{}{\xrightarrow{\text{CLL}} \Gamma \fatsemi \mathbf{1}}\ \mathbf{1}
\qquad
\dfrac{\xrightarrow{\text{CLL}} \Gamma \fatsemi \Delta, A, B}{\xrightarrow{\text{CLL}} \Gamma \fatsemi \Delta, A \parr B}\ \parr
\qquad
\dfrac{\xrightarrow{\text{CLL}} \Gamma \fatsemi \Delta}{\xrightarrow{\text{CLL}} \Gamma \fatsemi \Delta, \perp}\ \perp
$$

$$
\dfrac{\xrightarrow{\text{CLL}} \Gamma \fatsemi \Delta, A \quad \xrightarrow{\text{CLL}} \Gamma \fatsemi \Delta, B}{\xrightarrow{\text{CLL}} \Gamma \fatsemi \Delta, A \,\&\, B}\ \&
\qquad
\dfrac{}{\xrightarrow{\text{CLL}} \Gamma \fatsemi \Delta, \top}\ \top
$$

$$
\dfrac{\xrightarrow{\text{CLL}} \Gamma \fatsemi \Delta, A}{\xrightarrow{\text{CLL}} \Gamma \fatsemi \Delta, A \oplus B}\ \oplus_1
\qquad
\dfrac{\xrightarrow{\text{CLL}} \Gamma \fatsemi \Delta, B}{\xrightarrow{\text{CLL}} \Gamma \fatsemi \Delta, A \oplus B}\ \oplus_2
\qquad
\text{no rule for } \mathbf{0}
$$

$$
\dfrac{\xrightarrow{\text{CLL}} \Gamma, A \fatsemi \Delta}{\xrightarrow{\text{CLL}} \Gamma \fatsemi ?A, \Delta}\ ?
\qquad
\dfrac{\xrightarrow{\text{CLL}} \Gamma \fatsemi A}{\xrightarrow{\text{CLL}} \Gamma \fatsemi !A}\ !
$$

For arbitrary propositions A, the negation $A^{\perp}$ is defined using DeMorgan's laws, *i.e.*, $(A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp}$, *etc.* It is easy to see that $(A^{\perp})^{\perp} = A$ and $\xrightarrow{\text{CLL}} \cdot \fatsemi A^{\perp}, A$. We write $A \overset{\text{CLL}}{\equiv} B$ if both $\xrightarrow{\text{CLL}} \cdot \fatsemi A^{\perp}, B$ and $\xrightarrow{\text{CLL}} \cdot \fatsemi A, B^{\perp}$ are provable.

Structural properties of CLL (see [3, 13])

1. For any Γ and A, $\xrightarrow{\text{CLL}} \Gamma \fatsemi A^{\perp}, A$. (init′)
2. If $\xrightarrow{\text{CLL}} \Gamma \fatsemi \Delta$, then $\xrightarrow{\text{CLL}} \Gamma, A \fatsemi \Delta$ for any A. (!weak)

4.1 Double Negation Translation

The intuitionist views a classical proof of A as a refutation of its negation. In our setting, this would correspond to a proof of $\cdot \fatsemi [\![A]\!] \Longrightarrow \perp \fatsemi \cdot$, where $[\![A]\!]$ is the translation of A. It is more economical and allows some further applications if we instead parameterize the translation by a propositional parameter p and verify that $\cdot \fatsemi [\![A]\!]_p \Longrightarrow p \fatsemi \cdot$, an idea that goes back to Friedman [12] and was also employed by Lamarche [17] in the linear setting. It is convenient to introduce the *parametric negation* $\sim_p A = A \multimap p$, where $\sim_{\perp} A$ is the usual negation in JILL. Since the translation of A becomes a linear hypothesis, all connectives except $\otimes$, $\mathbf{1}$, and $!$ can simply be dualized. For the remaining three we need to introduce a double negation.

$$\begin{array}{rclcrcl}
[\![P]\!]_p & = & P & \qquad & [\![P^\perp]\!]_p & = & \sim_p P \\
[\![A \otimes B]\!]_p & = & \sim_p (\sim_p [\![A]\!]_p \otimes \sim_p [\![B]\!]_p) & & [\![\mathbf{1}]\!]_p & = & \sim_p \mathbf{1} \\
[\![A \parr B]\!]_p & = & [\![A]\!]_p \otimes [\![B]\!]_p & & [\![\perp]\!]_p & = & \mathbf{1} \\
[\![A \mathbin{\&} B]\!]_p & = & [\![A]\!]_p \oplus [\![B]\!]_p & & [\![\top]\!]_p & = & \mathbf{0} \\
[\![A \oplus B]\!]_p & = & [\![A]\!]_p \mathbin{\&} [\![B]\!]_p & & [\![\mathbf{0}]\!]_p & = & \top \\
[\![!A]\!]_p & = & \sim_p ! \sim_p [\![A]\!]_p & & [\![?A]\!]_p & = & ![\![A]\!]_p
\end{array}$$

We lift this definition to contexts of propositions by translating every proposition in the context. Unsurprisingly, this translation preserves provability.

Theorem 5 (Preservation). *If* $\xrightarrow{\text{CLL}} \Gamma \,;\, \Delta$, *then* $[\![\Gamma]\!]_p \,;\, [\![\Delta]\!]_p \Longrightarrow p \,;\, \cdot$ *for any propositional parameter* p.

Proof. By structural induction on the derivation $\mathcal{C} :: \xrightarrow{\text{CLL}} [\![\Gamma]\!]_p \,;\, [\![\Delta]\!]_p$. We highlight the non-trivial cases.

1. $\mathcal{C}$ a structural rule.

$$\mathcal{C} = \frac{}{\xrightarrow{\text{CLL}} [\![\Gamma]\!]_p \,;\, [\![P]\!]_p, [\![P^\perp]\!]_p}\ \text{init}$$

$$\begin{array}{ll}
[\![\Gamma]\!]_p \,;\, p \Longrightarrow p \,;\, \cdot & \text{by init} \\
[\![\Gamma]\!]_p \,;\, [\![P]\!]_p \Longrightarrow [\![P]\!]_p \,;\, \cdot & \text{by init} \\
[\![\Gamma]\!]_p \,;\, [\![P]\!]_p, [\![P^\perp]\!]_p \Longrightarrow p \,;\, \cdot & \text{by} \multimap L
\end{array}$$

The copy and promote rules are similar.

2. The last rule in $\mathcal{C}$ is $\otimes$.

$$\mathcal{C} = \frac{\xrightarrow{\text{CLL}} \Gamma \,;\, \Delta_1, A \qquad \xrightarrow{\text{CLL}} \Gamma \,;\, \Delta_2, B}{\xrightarrow{\text{CLL}} \Gamma \,;\, \Delta_1, \Delta_2, A \otimes B}\ \otimes$$

$$\begin{array}{ll}
[\![\Gamma]\!]_p \,;\, [\![\Delta_1]\!]_p, [\![A]\!]_p \Longrightarrow p \,;\, \cdot & \text{by i.h.} \\
[\![\Gamma]\!]_p \,;\, [\![\Delta_1]\!]_p \Longrightarrow \sim_p [\![A]\!]_p \,;\, \cdot & \text{by} \multimap R \\
[\![\Gamma]\!]_p \,;\, [\![\Delta_2]\!]_p \Longrightarrow \sim_p [\![B]\!]_p \,;\, \cdot & \text{similarly} \\
[\![\Gamma]\!]_p \,;\, [\![\Delta_1]\!]_p, [\![\Delta_2]\!]_p \Longrightarrow \sim_p [\![A]\!]_p \otimes \sim_p [\![B]\!]_p & \text{by} \otimes R \\
[\![\Gamma]\!]_p \,;\, p \Longrightarrow p \,;\, \cdot & \text{by init} \\
[\![\Gamma]\!]_p \,;\, [\![\Delta_1]\!]_p, [\![\Delta_2]\!]_p, [\![A \otimes B]\!]_p \Longrightarrow p \,;\, \cdot & \text{by} \multimap L
\end{array}$$

3. The last rule in $\mathcal{C}$ is $\perp$.

$$\mathcal{C} = \frac{\xrightarrow{\text{CLL}} \Gamma \,;\, \Delta}{\xrightarrow{\text{CLL}} \Gamma \,;\, \Delta, \perp}\ \perp$$

$$\begin{array}{ll}
[\![\Gamma]\!]_p \,;\, [\![\Delta]\!]_p, \Longrightarrow p \,;\, \cdot & \text{by i.h.} \\
[\![\Gamma]\!]_p \,;\, [\![\Delta]\!]_p, [\![\perp]\!]_p \Longrightarrow p \,;\, \cdot & \text{by } \mathbf{1}L
\end{array}$$

4. The last rule in $\mathcal{C}$ is $!$.

$$\mathcal{C} = \frac{\xrightarrow{\text{CLL}} \Gamma \,;\, A}{\xrightarrow{\text{CLL}} \Gamma \,;\, !A}\ !$$

$\llbracket \Gamma \rrbracket_p \mathbin{;} \cdot \Longrightarrow \sim_p \llbracket A \rrbracket_p$ by $\multimap R$ and i.h.
$\llbracket \Gamma \rrbracket_p \mathbin{;} \cdot \Longrightarrow \, ! \sim_p \llbracket A \rrbracket_p$ by $!R$
$\llbracket \Gamma \rrbracket_p \mathbin{;} p \Longrightarrow p \mathbin{;} \cdot$ by init
$\llbracket \Gamma \rrbracket_p \mathbin{;} \llbracket !A \rrbracket_p \Longrightarrow p \mathbin{;} \cdot$ by $\multimap L$

All other cases are trivial. □

By inspection of the proof we can see that the translation on classical derivations preserves their structure. Note that we do not need to employ ?, $\bot$, $\neg$, or $\parr$ in the image of the translation, a property achieved by introducing the parameter p.

The converse of the preservation theorem presents the more interesting property. We want to show *soundness* of the translation, *i.e.*, that provability of the image of this translation in JILL guarantees provability of the source in CLL. Such a statement cannot be shown by induction on the derivation of $\llbracket \Gamma \rrbracket_p \mathbin{;} \llbracket \Delta \rrbracket_p \Longrightarrow p \mathbin{;} \cdot$ because the JILL derivation may refer to propositions that are not in the image of the translation. Instead, we use the fact that classical linear logic admits more proofs on the same connectives: the reverse translation is simply the identity, where $A \multimap B$ has its classical definition as $A^\bot \parr B$. We write $\Delta^\bot$ to denote the classical negation of each proposition in Δ.

Theorem 6 (Reverse Preservation).

1. *If* $\Gamma \mathbin{;} \Delta \Longrightarrow A \mathbin{;} \cdot$, *then* $\xrightarrow{\text{CLL}} \Gamma^\bot \mathbin{;} \Delta^\bot, A$.
2. *If* $\Gamma \mathbin{;} \Delta \Longrightarrow \cdot \mathbin{;} A$, *then* $\xrightarrow{\text{CLL}} \Gamma^\bot, A \mathbin{;} \Delta^\bot$.

Proof. By examination of the last rule used in the derivation $\mathcal{D}_1 :: \Gamma \mathbin{;} \Delta \Longrightarrow A \mathbin{;} \cdot$ or $\mathcal{D}_2 :: \Gamma \mathbin{;} \Delta \Longrightarrow \cdot \mathbin{;} A$. We have the following cases.

1. init, *i.e.*,

$$\mathcal{D}_1 \;=\; \frac{}{\Gamma \mathbin{;} A \Longrightarrow A \mathbin{;} \cdot}\ \text{init}$$

$\xrightarrow{\text{CLL}} \Gamma^\bot \mathbin{;} A^\bot, A$ by init′

2. copy, *i.e.*,

$$\mathcal{D}_1 \;=\; \frac{\Gamma, B \mathbin{;} \Delta, B \Longrightarrow A \mathbin{;} \cdot}{\Gamma, B \mathbin{;} \Delta \Longrightarrow A \mathbin{;} \cdot}\ \text{copy}$$

$\xrightarrow{\text{CLL}} \Gamma^\bot, B^\bot \mathbin{;} \Delta^\bot, B^\bot, A$ by i.h.
$\xrightarrow{\text{CLL}} \Gamma^\bot, B^\bot \mathbin{;} \Delta^\bot, A$ by copy

3. promote, *i.e.*,

$$\mathcal{D}_1 \;=\; \frac{\Gamma \mathbin{;} \Delta \Longrightarrow A \mathbin{;} \cdot}{\Gamma \mathbin{;} \Delta \Longrightarrow \cdot \mathbin{;} A}\ \text{promote}$$

$\xrightarrow{\text{CLL}} \Gamma^\bot \mathbin{;} \Delta^\bot, A$ by i.h.
$\xrightarrow{\text{CLL}} \Gamma^\bot, A \mathbin{;} \Delta^\bot, A$ by !weak
$\xrightarrow{\text{CLL}} \Gamma^\bot, A \mathbin{;} \Delta^\bot$ by copy

4. a multiplicative right rule, say $\otimes R$.

$$\mathcal{D}_1 \;=\; \frac{\Gamma \mathbin{;} \Delta \Longrightarrow A \mathbin{;} \cdot \quad \Gamma \mathbin{;} \Delta' \Longrightarrow B \mathbin{;} \cdot}{\Gamma \mathbin{;} \Delta, \Delta' \Longrightarrow A \otimes B \mathbin{;} \cdot}\ \otimes R$$

$$\xrightarrow{\text{CLL}} \Gamma^\perp \mathbin{;} \Delta^\perp, A \qquad \text{by i.h.}$$
$$\xrightarrow{\text{CLL}} \Gamma^\perp \mathbin{;} {\Delta'}^\perp, B \qquad \text{by i.h.}$$
$$\xrightarrow{\text{CLL}} \Gamma^\perp \mathbin{;} \Delta^\perp, {\Delta'}^\perp, A \otimes B \qquad \text{by } \otimes$$

5. a multiplicative left rule, say $\multimap L$.

$$\mathcal{D}_1 = \frac{\Gamma \mathbin{;} \Delta \Longrightarrow B \mathbin{;} \cdot \quad \Gamma \mathbin{;} \Delta', D \Longrightarrow A \mathbin{;} \cdot}{\Gamma \mathbin{;} \Delta, \Delta', B \multimap D \Longrightarrow A \mathbin{;} \cdot} \multimap L$$

$$\xrightarrow{\text{CLL}} \Gamma^\perp \mathbin{;} \Delta^\perp, B \qquad \text{by i.h.}$$
$$\xrightarrow{\text{CLL}} \Gamma^\perp \mathbin{;} \Delta^\perp, D^\perp, A \qquad \text{by i.h.}$$
$$\xrightarrow{\text{CLL}} \Gamma^\perp \mathbin{;} \Delta^\perp, {\Delta'}^\perp, B \otimes D^\perp, A \qquad \text{by } \otimes$$

6. an additive rule, which is similar to the cases for the multiplicative rules.
7. an exponential right rule, say $!R$.

$$\mathcal{D}_1 = \frac{\Gamma \mathbin{;} \cdot \Longrightarrow A \mathbin{;} \cdot}{\Gamma \mathbin{;} \cdot \Longrightarrow !A \mathbin{;} \cdot} \; !R$$

$$\xrightarrow{\text{CLL}} \Gamma^\perp \mathbin{;} A \qquad \text{by i.h.}$$
$$\xrightarrow{\text{CLL}} \Gamma^\perp \mathbin{;} !A \qquad \text{by } !$$

8. an exponential left rule, say $?L$.

$$\mathcal{D}_2 = \frac{\Gamma \mathbin{;} B \Longrightarrow \cdot \mathbin{;} A}{\Gamma \mathbin{;} ?B \Longrightarrow \cdot \mathbin{;} A} \; ?L$$

$$\xrightarrow{\text{CLL}} \Gamma^\perp, A \mathbin{;} B^\perp \qquad \text{by i.h.}$$
$$\xrightarrow{\text{CLL}} \Gamma^\perp, A \mathbin{;} !B^\perp \qquad \text{by } !$$

All other cases are straightforward. □

The reverse translation is an inverse, but only up to classical equivalence.

Theorem 7 (Equivalence). *For any proposition A, if $p \overset{\text{CLL}}{\equiv} \bot$, then $(\llbracket A \rrbracket_p)^\perp \overset{\text{CLL}}{\equiv} A$.*

Proof. By structural induction on the proposition A. The interesting cases are the double-negation cases, for which the assumption $p \overset{\text{CLL}}{\equiv} \bot$ is required.

$$(\llbracket B \otimes C \rrbracket_p)^\perp = \left((\llbracket B \rrbracket_p)^\perp \parr p\right) \otimes \left((\llbracket C \rrbracket_p)^\perp \parr p\right) \otimes p^\perp \overset{\text{CLL}}{\equiv} (\llbracket B \rrbracket_p)^\perp \otimes (\llbracket C \rrbracket_p)^\perp$$
$$(\llbracket \mathbf{1} \rrbracket_p)^\perp = \mathbf{1} \otimes p^\perp \overset{\text{CLL}}{\equiv} \mathbf{1}$$
$$(\llbracket !B \rrbracket_p)^\perp = \left(!\,(\llbracket B \rrbracket_p)^\perp \parr p\right) \otimes p^\perp \overset{\text{CLL}}{\equiv} !\,(\llbracket B \rrbracket_p)^\perp$$
□

Soundness of the interpretation now follows directly from the previous two theorems.

Theorem 8 (Soundness). *If $\llbracket \Gamma \rrbracket_p \mathbin{;} \llbracket \Delta \rrbracket_p \Longrightarrow p \mathbin{;} \cdot$ with p a propositional parameter, then $\xrightarrow{\text{CLL}} \Gamma \mathbin{;} \Delta$.*

Proof. From Thm. 6, we know that $\xrightarrow{\text{CLL}} (\llbracket \Gamma \rrbracket_p)^\perp \mathbin{;} (\llbracket \Delta \rrbracket_p)^\perp, p$. By instantiating the parameter $p = \bot$ and using the admissibility of cut with $\xrightarrow{\text{CLL}} \cdot \mathbin{;} 1$, we get $\xrightarrow{\text{CLL}} (\llbracket \Gamma \rrbracket_\bot)^\perp \mathbin{;} (\llbracket \Delta \rrbracket_\bot)^\perp$. To complete the proof, we just appeal to Thm. 7. □

4.2 Exploiting Parametricity

By design, the $[\![-]\!]_p$ translation works even in the absence of linear contradiction on the intuitionistic side, but we already know that JILL allows a definition of contradiction $\bot$ as $?\mathbf{0}$. Using $p = \bot$ we obtain an elegant characterization of CLL with our interpretation as a consequence of the above theorems.

Theorem 9 (Characterizing CLL). $\xrightarrow{CLL} \Gamma \mathbin{⨾} \Delta$ *iff* $[\![\Gamma]\!]_\bot \mathbin{⨾} [\![\Delta]\!]_\bot \Longrightarrow \bot \mathbin{⨾} \cdot$.

Proof. From Theorems 5 and 6 as in the proof of Theorem 8. □

Of course, not all choices for p give the same behavior. Remarkably, choosing $p = \mathbf{1}$ produces an interpretation of CLL with the following additional MIX rules, first considered by Girard [13] and since attracted considerable interest.

$$\frac{}{\xrightarrow{\mathrm{CLL}^+} \Gamma \mathbin{⨾} \cdot}\ \mathsf{MIX}_0 \qquad \frac{\xrightarrow{\mathrm{CLL}^+} \Gamma \mathbin{⨾} \Delta \quad \xrightarrow{\mathrm{CLL}^+} \Gamma \mathbin{⨾} \Delta'}{\xrightarrow{\mathrm{CLL}^+} \Gamma \mathbin{⨾} \Delta, \Delta'}\ \mathsf{MIX}_2$$

We write CLL$^+$ for CLL with these MIX rules. Like Thm. 9, $[\![-]\!]_{\mathbf{1}}$ characterizes CLL$^+$.

Theorem 10 (Characterizing CLL$^+$). $\xrightarrow{CLL^+} \Gamma \mathbin{⨾} \Delta$ *iff* $[\![\Gamma]\!]_{\mathbf{1}} \mathbin{⨾} [\![\Delta]\!]_{\mathbf{1}} \Longrightarrow \mathbf{1} \mathbin{⨾} \cdot$.

Proof. For the forward direction, we note the admissibility of the following rule in JILL, by using $\mathbf{1}L$ on the second premiss and admissibility of cut (Thm. 2).

$$\frac{\Gamma \mathbin{⨾} \Delta \Longrightarrow \mathbf{1} \mathbin{⨾} \cdot \quad \Gamma \mathbin{⨾} \Delta' \Longrightarrow \mathbf{1} \mathbin{⨾} \cdot}{\Gamma \mathbin{⨾} \Delta, \Delta' \Longrightarrow \mathbf{1} \mathbin{⨾} \cdot}$$

We use preservation (Thm. 5) for all rules of CLL, and $\mathbf{1}R$ and the above admissible rule for the MIX rules. For the reverse direction, we use Thm. 6 and admissibility of cut with $\xrightarrow{\mathrm{CLL}^+} \cdot \mathbin{⨾} \bot$ to conclude that $\xrightarrow{\mathrm{CLL}^+} ([\![\Gamma]\!]_{\mathbf{1}})^\bot \mathbin{⨾} ([\![\Delta]\!]_{\mathbf{1}})^\bot$. Now note that $([\![A]\!]_{\mathbf{1}})^\bot \overset{\mathrm{CLL}^+}{\equiv} A$ for any A by Thm. 7 and the collapse $\mathbf{1} \overset{\mathrm{CLL}^+}{\equiv} \bot$ in the presence of the MIX rules. □

Therefore, a proof in CLL$^+$ corresponds to a JILL proof of $\mathbf{1}$. In other words, a classical proof using the MIX rules can be seen as an intuitionistic proof that can consume all resources. This analysis interprets $A^\bot$ as a consumer of A, that is, $A \multimap \mathbf{1}$. This finally provides a clear understanding of the MIX rules from a logical and constructive viewpoint. Note that, classically, this explanation does not work, because $(A \multimap \mathbf{1}) \multimap \mathbf{1}$ is *not* equivalent to A.

The remarkable uniformity of the parametric translation $[\![-]\!]_p$ demands the question: what about other choices for p? A full examination lies beyond the scope of this paper; however, at least one further result is easy to obtain: choosing $p = \mathbf{0}$ so that $\sim_p A = A \multimap \mathbf{0}$ interprets CLL with the following additional weakening rule, *i.e.*, classical affine logic (CAL).

$$\frac{\xrightarrow{\mathrm{CAL}} \Gamma \mathbin{⨾} \Delta}{\xrightarrow{\mathrm{CAL}} \Gamma \mathbin{⨾} \Delta, A}\ \mathsf{weak}$$

Theorem 11 (Characterizing CAL). $\xrightarrow{CAL} \Gamma \mathbin{⨾} \Delta$ *iff* $[\![\Gamma]\!]_{\mathbf{0}} \mathbin{⨾} [\![\Delta]\!]_{\mathbf{0}} \Longrightarrow \mathbf{0} \mathbin{⨾} \cdot$.

Proof. For the forward direction, we use preservation (Thm. 5) for all rules of CLL, and the following additional admissible rule in JILL:

$$\frac{\Gamma \mathbin{⨾} \Delta \Longrightarrow \mathbf{0} \mathbin{⨾} \cdot}{\Gamma \mathbin{⨾} \Delta, A \Longrightarrow \mathbf{0} \mathbin{⨾} \cdot}$$

(Use cut-admissibility (Thm. 2) with cut-formula $\mathbf{0}$ and the evident sequent $\Gamma \mathbin{;} A, \mathbf{0} \Longrightarrow \mathbf{0} \mathbin{;} \cdot$.)

For the reverse direction, we use Thm. 6 and admissibility of cut with $\xrightarrow{\text{CAL}} \cdot \mathbin{;} \top$ to conclude that $\xrightarrow{\text{CAL}} (\llbracket \Gamma \rrbracket_{\mathbf{0}})^{\perp} \mathbin{;} (\llbracket \Delta \rrbracket_{\mathbf{0}})^{\perp}$, and then use Thm. 7 with the characteristic collapse $\mathbf{1} \overset{\text{CAL}}{\equiv} \top$ (dually $\mathbf{0} \overset{\text{CAL}}{\equiv} \perp$) of affine logic. □

5 Conclusion and Future Work

We have presented a formulation of intuitionistic linear logic with a clear distinction between judgments and propositions. In this system, the full range of additive, multiplicative, and exponential connectives seen in classical linear logic—including ?, $\perp$, $\neg$, and $\parr$—admit definitions either directly with introduction and elimination rules or as notational definitions. The judgmental treatment allows the development of a sequent calculus with a structural cut-admissibility proof, which formalizes Martin-Löf's notion of *verification* as cut-free proofs in the sequent calculus. By design, the sequent calculus has a natural bottom-up reading, and we can prove the admissibility of cut in a straightforward manner. Using our sequent calculus, we develop a parametric interpretation of classical linear logic and give a constructive interpretation of the structural MIX rules.

Term calculi for the natural deduction formulation of JILL and their computational interpretation provides an interesting area for future work. For example, a natural interpretation of $A \multimap ?B$ allows linear partial functions, while $\parr$ and $\perp$ may be related to concurrency. Finally, the nature of possibility requires a categorical explanation as a dual of the standard comonad constructions for !.

For the parametric interpretations of CLL, one interesting topic for future work is an exploration of similar embeddings of other substructural logics—for example, FILL, classical strict logic, or even JILL itself—into JILL.

Acknowledgments. We would like to thank Jeff Polakow for valuable discussions regarding the judgmental nature of linear logic and Valeria de Paiva for helping us understand full intuitionistic linear logic.

References

1. S. Abramsky. Computational interpretations of linear logic. *Theoretical Computer Science*, 111:3–57, 1993.
2. Klaus Aehlig and Helmut Schwichtenberg. A syntactical analysis of non-size-increasing polynomial time computation. *Submitted*, 2001. A previous version presented at LICS'00.
3. J.-M. Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):297–347, 1992.
4. Andrew Barber. Dual Intuitionistic Linear Logic. Technical Report ECS-LFCS-96-347, University of Edinburgh, 1997.
5. Josh Berdine, Peter O'Hearn, Uday S. Reddy, and Hayo Thielecke. Linear continuation-passing. *Higher-Order and Symbolic Computation*, 15:181–208, September 2002.
6. G. Bierman. *On Intuitionistic Linear Logic*. PhD thesis, University of Cambridge, 1994.
7. Gavin M. Bierman. A note on full intuitionistic linear logic. *Annals of Pure and Applied Logic*, 79(3):281–287, 1996.
8. Torben Braüner and Valeria de Paiva. A formulation of linear logic based on dependency-relations. In Mogens Nielsen and Wolfgang Thomas, editors, *Proceedings of the Annual Conference of the European Association for Computer Science Logic*, volume 1414 of *LNCS*, pages 129–148, Aarhus, Denmark, August 1997. Springer Verlag.
9. Iliano Cervesato and Frank Pfenning. A linear logical framework. *Information & Computation*, 179(1):19–75, November 2002.

10. J. Robin B. Cockett and R.A.G. Seely. Proof theory for full intuitionistic linear logic, bilinear logic and MIX categories. *Theory and Applications of Categories*, 3:85–131, 1997.
11. K. Došen. Modal translations in substructural logics. *Journal of Philosophical Logic*, 21:283–336, 1992.
12. Harvey Friedman. Classically and intuitionistically provably recursive functions. In D.S. Scott and G.H. Muller, editors, *Higher Set Theory*, pages 21–27. Springer-Verlag LNM 699, 1978.
13. J.-Y. Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.
14. J. S. Hodas and D. Miller. Logic programming in a fragment of intuitionistic linear logic. *Information and Computation*, 110(2):327–365, 1994. Extended abstraction in the Proceedings of the Sixth Annual Symposium on Logic in Computer Science, Amsterdam, July 15–18, 1991.
15. Martin Hofmann. Linear types and non-size increasing polynomial time computation. *Theoretical Computer Science*, 2000. To appear. A previous version was presented at LICS'99.
16. Martin Hyland and Valeria de Paiva. Full intuitionistic linear logic (extended abstract). *Annals of Pure and Applied Logic*, 64(3):273–291, 1993.
17. François Lamarche. Games semantics for full propositional linear logic. In D. Kozen, editor, *Proceedings of the 10th Annual Symposium on Logic in Computer Science (LICS'95)*, pages 464–473, San Diego, California, June 1995. IEEE Computer Society Press.
18. I. Mackie. Lilac — a functional programming language based on linear logic. *Journal of Functional Programming*, 4(4):395–433, 1993.
19. Per Martin-Löf. On the meanings of the logical constants and the justifications of the logical laws. *Nordic Journal of Philosophical Logic*, 1(1):11–60, 1996. Lectures notes to a short course at Università degli Studi di Siena, April 1983.
20. Michel Parigot. $\lambda\mu$-calculus: An algorithmic interpretation of classical natural deduction. In A. Voronkov, editor, *Proceedings of the International Conference on Logic Programming and Automated Reasoning*, pages 190–201, St. Petersburg, Russia, July 1992. Springer-Verlag LNCS 624.
21. Frank Pfenning. Structural cut elimination in linear logic. Technical Report CMU-CS-94-222, Department of Computer Science, Carnegie Mellon University, December 1994.
22. Frank Pfenning. Structural cut elimination I. intuitionistic and classical logic. *Information and Computation*, 157(1/2):84–141, March 2000.
23. Frank Pfenning and Rowan Davies. A judgmental reconstruction of modal logic. *Mathematical Structures in Computer Science*, 11(4):511–540, 2001.
24. Jeff Polakow. *Ordered Linear Logic and Applications*. PhD thesis, Department of Computer Science, Carnegie Mellon University, August 2001.
25. Jeff Polakow and Frank Pfenning. Relating natural deduction and sequent calculus for intuitionistic non-commutative linear logic. In Andre Scedrov and Achim Jung, editors, *Proceedings of the 15th Conference on Mathematical Foundations of Programming Semantics*, New Orleans, Louisiana, April 1999. Electronic Notes in Theoretical Computer Science, Volume 20.
26. Eike Ritter, David Pym, and Lincoln Wallen. On the intuitionistic force of classical search. *Theoretical Computer Science*, 232(1–2):299–333, 2000.
27. Wilfried Sieg and John Byrnes. Normal natural deduction proofs (in classical logic). *Studia Logica*, 60(1):67–106, January 1998.
28. P. Wadler. A syntax for linear logic. In S. Brookes, M. Main, A. Melton, M. Mislove, and D. Schmidt, editors, *Proceedings of the Ninth Conference on Mathematical Foundations of Programming Semantics*, pages 513–529, New Orleans, Louisiana, April 1993. Springer-Verlag LNCS 647.