Exploring Variations of Wythoff Nim

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For my carries

Abstract

We investigate two modifications to the traditional rules of Wythoff Nim, a combinatorial game. In Wythoff Nim, players take turns removing stones from a pair of piles. In each turn, a player chooses to either (1) take any number of stones from one pile or (2) take an equal number from both. The player removing the last stone wins. A seminal result of Wythoff states that at any point in the game, the current player is in a P-position — that is, guaranteed to lose assuming the other player plays optimally — if and only if the pair of pile sizes is of the form $(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor)$ for some $n \in \mathbb{N}$, where $\phi = 1.618\ldots$ represents the golden ratio.

This thesis introduces a variant of Wythoff Nim, which we call $W(a,b)$, characterized by positive integers a and b, where players' moves are constrained to either removing a multiple of a from the first pile, a multiple of b from the second, or an equal number from both. We prove that the P-positions of this variant also follow sloping "beams", but with slopes determined by a and b.

Additionally, we explore the consequences of modifying Wythoff's Nim by introducing additional fixed winning and losing positions. Our empirical observations suggest a convergence of the P-positions in the long run, resembling those of the regular Wythoff game. Together, these results shed light on the applicability of Wythoff's theory to variants of his original game.

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Chapter 1

Introduction

Combinatorial games are two-player games characterized by perfect information[1] and the fully deterministic gameplay, resulting in a win or loss outcome in a finite number of moves. They provide a rich area of study across disciplines such as mathematics, computer science, game theory, and recreational gaming. A combinatorial game can be described by the set of positions in the game, the set of rules dictating permissible moves, the winning condition, and the initial position. The first player makes the initial move, after which the two players alternate turns until reaching a *terminal position*, where no additional moves are possible. At this point, the last player to make a move is either the winner or the loser, depending on whether the game is *normal play* or *misère play*, respectively. Central to this field are *impartial games*, wherein the same set of moves is available to both players from any position. [2]

More precisely, a game is a combinatorial game if it satisfies the following [3, 6]:

1. Two players alternate moving.
2. There are finitely many possible positions in the game.
3. The game ends when one player has no legal moves left, which is also referred to as a terminal position. Furthermore, the game must end after a finite number of moves.
4. Under normal play, the last player to move wins; under misère play, the last player to move loses. There are no draws in the game.
5. The rules specify, for each player and for each position, which moves to other positions are legal.

Note that there are three conditions inherent in this definition: no randomness, no hidden states, and no draws. For example, games that involve rolling dice are not combinatorial games. Both players should have perfect information; therefore, we also rule out games like poker and rock-paper-scissors. Furthermore, there should be one clear winner when the game terminates, which rules out Chess. In this thesis, we will focus on impartial games. Impartial games are a particularly simple and important subclass of combinatorial games, where players face identical

[1] That is, the entire state of the game is known to both players at all times.

[2] Conversely, a combinatorial game is *partisan* if the two players have different sets of possible moves from a given position. For instance, Chess is a partisan game since the players control different sets of pieces and have distinct options for moves based on their positions on the board.

options at every turn, with no inherent advantage or disadvantage between them. *Nim* is a classic example of an impartial combinatorial game. Its rules are straightforward: players take turns removing objects from distinct piles. On each turn, a player must remove at least one object, and they can take as many objects as they wish from a single pile. The player who removes the last object wins.

Our goal is to rigorously analyze possible winning strategies in the game of Nim. That is, we aim to distinguish between states where it is possible to win and states where it is not. In combinatorial game theory, we employ the concepts of *P-positions* and *N-positions* to categorize the winning and losing scenarios within a game. Unless otherwise specified, the games we analyze are played in the normal version rather than the misère version. A P-position, also known as a losing position, occurs when the current player has no means of winning the game. In such a position, regardless of the current player's actions, the next player will always possess a winning strategy for their subsequent turn. Conversely, an N-position, alternatively referred to as a winning position, provides the current player with a winning strategy. This implies there is some move they can make to force their opponent into a losing position.

In the context of one-pile Nim, an empty pile is a terminal state, marking it as a losing position or a P-position, whereas a pile containing objects is an N-position, as a player can win by depleting the pile entirely. The P-positions of two-pile Nim are piles of equal size. Specifically, pairs of piles with sizes (0, 0), (1, 1), (2, 2), and so forth, are considered P-positions [6]. In these configurations, the second player to move can always mirror the first player's actions, ensuring that the piles remain equal after each pair of turns. This symmetry leads to a winning strategy for the player controlling these positions. Conversely, configurations where the piles have unequal sizes are considered N-positions. In the case of multi-pile Nim, we employ the concept of *Nimber* to formulate a successful strategy, which will be elaborated upon in the subsequent section.

In our examination of Nim and its variants, *Wythoff Nim* emerges as a notable iteration. Stemming from a theorem by W. A. Wythoff [7], this variant introduces complexity by allowing players to manipulate two piles of objects simultaneously. In Wythoff Nim, players face two piles of objects and can remove any number of objects from a single pile or an equal number from both piles. The P-positions in Wythoff Nim differ significantly from those in regular 2-pile Nim. In particular, two equally sized piles are no longer a P-position. It turns out that the P-positions of Wythoff Nim follow a distinct pattern and are associated with the golden ratio.

In this thesis, our focus extends beyond traditional Nim as we delve into variations of Wythoff Nim. We explore how slight alterations in the rules of each round can impact winning strategies, comparing these variations to the classic Wythoff Nim to uncover common winning patterns. Additionally, we investigate another intriguing variation where we declare a finite number of positions to be P-positions initially, examining its asymptotic behavior in the long term. Through these analyses, we aim to deepen our understanding of strategic gameplay in Wythoff Nim and its variants, as well as the underlying principles that govern optimal decision-making in impartial combinatorial games.

Chapter 2

Related Work

In this section, we begin by introducing the fundamental principles of the game of Nim and Wythoff Nim, highlighting its unique rules and strategic intricacies. Moreover, we explore various characterizations of winning positions within Wythoff Nim. By examining the existing body of work on Wythoff Nim and its winning strategies, we aim to contextualize our own contributions to understanding this variant and its implications for combinatorial game theory.

2.1 The Game of Nim

P-positions and *N-positions* are commonly associated with impartial games, where both players have identical options available to them at any given point. In this section, we will formally define these terms to facilitate our analysis of the winning strategies in games such as Nim and Wythoff Nim.

Definition 1 (P-positions and N-positions, e.g. [6])**.** *In a combinatorial game, the following recursive procedure identifies every state uniquely as either a* P-position *or* N-position*:*

1. *All terminal positions are P-positions (normal winning rule).*
2. *A position where all moves give N-positions is an P-position.*
3. *A position where at least 1 move gives a P-position is an N-position.*

Determining whether a given position is a P-position or an N-position is crucial for analyzing combinatorial games and devising winning strategies. By identifying the winning and losing positions, players can make informed decisions to steer the game towards a favorable outcome.

The terms P- and N-position indicate which player can win given a game state in between two player's turns. P-positions are positions that are winning for the Previous player (the player who just moved) (Sometimes called LOSING positions). N-positions are the positions that are winning for the Next player (the player who is about to move) (Sometimes called WINNING positions).

In the previous section, we examined the P-positions of one-pile and two-pile Nim, finding that they align with the recursive definitions provided earlier. However, determining the P-positions of Nim with three or more piles presents an additional challenge, requiring us to utilize the concept of Nimber.

Theorem 1 (Bouton [2]). *A position (x, y, z) in Nim is a P-position if and only if $x \oplus y \oplus z$ = 0, where $\oplus$ represents the bitwise XOR (exclusive or) operation.*

The Bouton Theorem also extends its applicability to Nim games with a pile size of 4 or greater[1]. The table below displays a subset of P-positions in 3-pile Nim. It is noteworthy that when one of the piles has a size of 0, the game simplifies to a standard 2-pile Nim, and the P-positions occur precisely when the remaining two piles have equal sizes.

Table 2.1: P-positions of regular 3-pile Nim

(0, m, n)	**(1, m, n)**	**(2, m, n)**	**(3, m, n)**	**(4, m, n)**	**(5, m, n)**
(0, 0, 0)	(1, 2, 3)	(2, 4, 6)	(3, 4, 7)	(4, 8, 12)	(5, 8, 13)
(0, 1, 1)	(1, 4, 5)	(2, 5, 7)	(3, 5, 6)	(4, 9, 13)	(5, 9, 12)
(0, 2, 2)	(1, 6, 7)	(2, 8, 10)	(3, 8, 11)	(4, 10, 14)	(5, 10, 15)
(0, 3, 3)	(1, 8, 9)	(2, 9, 11)	(3, 9, 10)	(4, 11, 15)	(5, 11, 14)
(0, 4, 4)	(1, 10, 11)	(2, 12, 14)	(3, 12, 15)	(4, 16, 20)	(5, 16, 21)
(0, 5, 5)	(1, 12, 13)	(2, 13, 15)	(3, 13, 14)	(4, 17, 21)	(5, 17, 20)
(0, 6, 6)	(1, 14, 15)	(2, 16, 18)	(3, 16, 19)	(4, 18, 22)	(5, 18, 23)
(0, 7, 7)	(1, 16, 17)	(2, 17, 19)	(3, 17, 18)	(4, 19, 23)	(5, 19, 22)
(0, 8, 8)	(1, 18, 19)	(2, 20, 22)	(3, 20, 23)	(4, 24, 28)	(5, 24, 29)
(0, 9, 9)	(1, 20, 21)	(2, 21, 23)	(3, 21, 22)	(4, 25, 29)	(5, 25, 28)
(0, 10, 10)	(1, 22, 23)	(2, 24, 26)	(3, 24, 27)	(4, 26, 30)	(5, 26, 31)

Proof (adapted from [6]). Let's check the recursive definitions for P-positions and N-positions. The nim-sum of two non-negative integers is their addition without carry in base 2. Equivalently, we can write each of the two non- negative integer in their binary form and compute their nim-sum by XORing them. Note that the operation is associative and commutative.

1. All terminal positions are P-positions. In the game, there is only one terminal position: when all piles are empty. This position has a nim-sum of 0 because $0 \oplus 0 \oplus 0 \oplus ... \oplus 0 = 0$.
2. A position where all moves give N-positions is an P-position. Suppose the current game state $(x_1, x_2, .., x_n)$ is a P-position, meaning $x_1 \oplus x_2 \oplus .... \oplus x_n = 0$. When the next player takes from a single pile, let's say x_1, and reduces it to x'_1. The cancellation property states that $x \oplus y = x \oplus z \Rightarrow y = z$. Since $x'_1 \oplus x_2 \oplus .... \oplus x_n \neq 0$ (otherwise $x_1 = x'_1$), the new state $(x'_1, x_2, .., x_n)$ is an N-position.
3. A position where at least 1 move gives a P-position is an N-position. Suppose $(x_1, x_2, .., x_n)$ is an N-position. We represent each x_i in binary and use column addition when computing the nim-sum. We examine the leftmost column with an odd number of 1's. We can change any number with a 1 in that column so that the nim-sum becomes 0, a P-position. This change is equivalent to decreasing from that pile, as the leftmost 1 is changed to 0.

□

[1]Proven in the Sprague-Grundy Theorem

Remark 1 (Strategies for Misère Nim)**.** *According to Bouton's Theorem, in a standard N-pile Nim game, a player can secure their win by ensuring that the XOR of the piles sizes equals 0 after each of their turns. This strategy remains effective most of the time even in the misère version of the game. When there are multiple piles each with a size of at least 2, the player employs the same strategy as in the regular game, aiming to make the nim-sum 0 after each turn. However, if there is only one pile with a size of at least 2, the player can reduce that pile to 0 or 1, thus creating an odd number of piles with a size of 1. This odd configuration constitutes a P-position, forcing the next player to take the last piece.*

This strategy is effective because after the player leaves the nim-sum as 0, there must be at least two piles with a size of at least 2, eventually leading the game to a point where there is only one pile of size at least 2.

In comparison to the regular Nim game, the P-positions remain identical in the misère version when there is at least one pile with a size of at least 2. However, when all piles have a size of 0 or 1, the P-positions in the misère game correspond to the N-positions in the regular game, and vice versa.

2.2 Two Characterization of Wythoff Nim's P-positions

Here is the original game rules defined by Wythoff from his 1907 paper [7]:

> "The game is played by two persons. Two piles of counters are placed on the table, the number of each pile being arbitrary. The players play alternately and either take from one of the piles an arbitrary number of counters or from both piles an equal number. The player who takes up the last counter or counters, wins."

Since the rules of Wythoff Nim are symmetric across both piles, the P-positions exhibit symmetry as well, meaning that (x, y) is a P-position if and only if (y, x) is also a P-position. Later, we'll observe that the P-positions form two lines originating from the origin. The table below illustrates the numerical values of P-positions of Wythoff Nim on the upper line. The nth P-position is denoted by (x_n, y_n).

Table 2.2: Values of n, x_n, and y_n in Wythoff's game

n	0	1	2	3	4	5	6	7	8	9	10	11	12
x_n	0	1	3	4	6	8	9	11	12	14	16	17	19
y_n	0	2	5	7	10	13	15	18	20	23	26	28	31

Note that the difference between x_n and y_n is n. Together with the following notion, this will give us our first characterization of the P-positions of Wythoff Nim.

Definition 2 (Minimum excluded element)**.** *Let S be a set of natural numbers. The* minimum excluded element *of S, denoted $\mathsf{MEX}(S)$, is the smallest non-negative integer that is not present in the set.*

For instance, consider the set $\{0, 1, 2, 4, 5, 6\}$. The MEX of this set is 3, as it is the smallest non-negative integer absent from the set. Similarly, the MEX of $\{1, 3, 5, 7, 9\}$ is 0, and for an empty set $\{\}$, the MEX is also 0.

Theorem 2 (Recursive characterization of P-positions in Wythoff Nim [7])**.** *Let* $(x_0, y_0) = (0, 0)$*. Inductively, for* $n \in \mathbb{Z}^+$*, let* $x_n = \mathsf{MEX}(\{x_k : 0 \leq k < n\} \cup \{y_k : 0 \leq k < n\})$ *and* $y_n = x_n + n$*. Then the P-positions in Wythoff Nim are precisely* $\{(x_n, y_n) : n \in \mathbb{N}\} \cup \{(y_n, x_n) : n \in \mathbb{N}\}$.

Theorem 2 claims that following the terminal position (0, 0), subsequent P-positions can be constructed based on preceding ones. The nth P-position on the upper line, denoted as (x_n, y_n), is defined by setting x_n as the MEX of the set comprising both coordinates from preceding P-positions, with y_n calculated as $x_n + n$.

Proof (adapted from [7]). To show that the P-positions can be generated this way, we check the recursive definitions for P-positions and N-positions.

1. All terminal positions are P-positions. The only terminal position is (0, 0), serving as the base case for the recursive characterization of P-positions.
2. A position where all moves give N-positions is a P-position. Let (x_n, y_n) denote the nth P-position generated by the recursive scheme. Considering all possible moves from (x_n, y_n), if any lead to a P-position, they must be of the form (x_k, y_k) or (y_k, x_k) for some $k < n$. Since $y_k - x_k = k < n$, removing an equal number from both piles cannot result in a P-position. As x_n is defined as the minimum excluded element from the coordinates of previous P-positions, it strictly increases with each n. Similarly, the gap between y_n and x_n grows larger with each iteration, leading to a strict increase in y_n as n increases. This also demonstrates that x_k, y_k, x_n, and y_n are all distinct. Therefore, transitioning from (x_n, y_n) to (x_k, y_k) by removing objects from a single pile is impossible.
3. A position where at least 1 move gives a P-position is an N-position. Let (x, y) represent a position that is not a P-position, with $x \leq y$. Define $n = y - x$. Note that x cannot equal x_n, otherwise (x, y) would be a P-position. If $x > x_n$, subtracting $x - x_n$ from both piles results in $(x_n, y - x + x_n) = (x_n, n + x_n) = (x_n, y_n)$, which is a P-position. If $x < x_n$, then by definition there exists some $k < n$ such that either $x = x_k$ or $x = y_k$. If $x = x_k$, removing $y - y_k$ from the second pile yields (x_k, y_k). Similarly, if $x = y_k$, removing $y - x_k$ from the second pile yields (y_k, x_k), which is also a P-position. Thus, there always exists a move from (x, y) leading to a P-position.

□

Figure 2.1 illustrates the P-positions of Wythoff Nim, where the x-coordinate denotes the number of objects in the first pile and the y-coordinate represents the number of objects in the second pile. It's important to note that the grid is 0-indexed, starting with the first P-position at $(0, 0)$. Due to the symmetric rules of Wythoff Nim, the P-positions are mirrored across the diagonal line $y = x$. Additionally, the P-positions form distinct patterns resembling two beams emanating from the origin. These beams have slopes of ϕ and $\frac{1}{\phi}$ respectively, where ϕ represents the golden ratio $\frac{1+\sqrt{5}}{2}$.

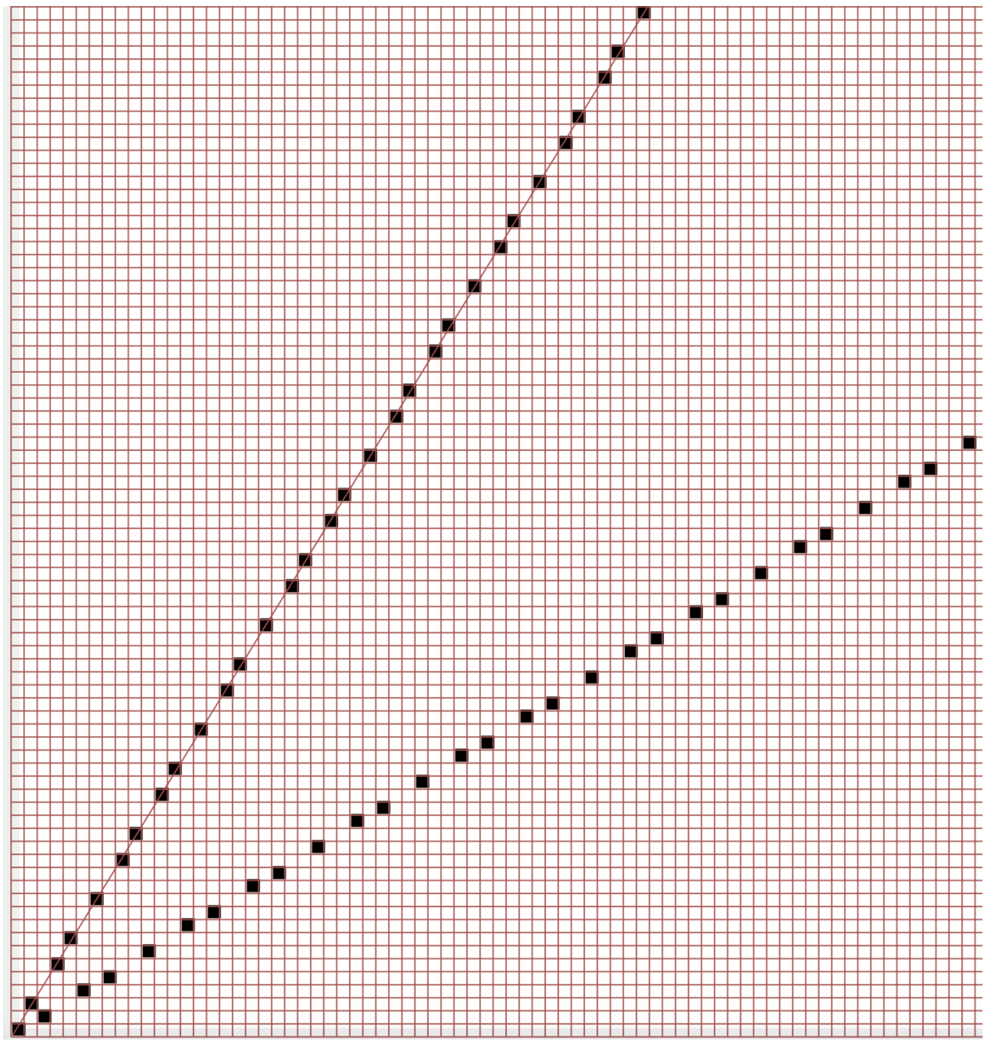

Figure 2.1: P-positions of Wythoff Nim

This graphical representation brings us to the second characterization of P-positions of Wythoff Nim.

Theorem 3 (Algebraic characterization of P-positions in Wythoff Nim [7])**.** *The P-positions of Wythoff Nim are precisely the points* $\{(\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor) : n \in \mathbb{N}\} \cup \{(\lfloor n\phi^2 \rfloor, \lfloor n\phi \rfloor) : n \in \mathbb{N}\}$*, where* $\phi = \frac{1+\sqrt{5}}{2}$.

Theorem 3 asserts that we can systematically enumerate the P-positions in a Wythoff game by moving rightward and upward, effectively traversing a grid. This method captures all P-positions on the upper line. Notably, each step to the right increases the value of $\lfloor n\phi \rfloor$ by either 1 or 2, corresponding to $\lfloor \phi \rfloor$ and $\lceil \phi \rceil$, respectively. Similarly, each step in the positive y-direction increments by 2 or 3, governed by $\lfloor \phi^2 \rfloor$ and $\lceil \phi^2 \rceil$. This systematic approach stems from the game's inherent symmetry. Consequently, the P-positions on the lower line can be represented as $(\lfloor n\phi^2 \rfloor, \lfloor n\phi \rfloor)$. With this algebraic formulation, the P-positions align along the lines $y = \phi x$ and $y = \frac{1}{\phi}x$, as depicted in the figure.

Proof (adapted from [7]). To prove Theorem 3, we establish a bijection between the algebraic characterizations presented here and the recursive characterizations outlined in Theorem 2. Specifically, let (x_n, y_n) denote the sequence of P-positions in the statement of Theorem 3. We prove that $(x_n, y_n) = (\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor)$.

For the base case, when $n = 0$, both $\lfloor n\phi \rfloor$ and $\lfloor n\phi^2 \rfloor$ equal 0, representing the terminal P-position $(x_0, x_0) = (0, 0)$.

For $n > 0$, we observe:

$$\begin{aligned} \lfloor n\phi^2 \rfloor - \lfloor n\phi \rfloor &= \lfloor n(\phi + 1) \rfloor - \lfloor n\phi \rfloor \\ &= n + \lfloor n\phi \rfloor - \lfloor n\phi \rfloor = n \end{aligned}$$

This equation demonstrates a consistent gap of n between successive P-positions along the x-axis.

Now, we aim to show that $\lfloor n\phi \rfloor$ is the Minimum Excluded Element (MEX) of the set formed by combining the sequences $\{\lfloor k\phi \rfloor : k < n\}$ and $\{\lfloor k\phi^2 \rfloor : k < n\}$. We need the following well-known theorem:

Theorem 4 (Beatty [1])**.** *If α and β are irrational numbers such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then the sequences $\{\lfloor m\alpha \rfloor : m \in \mathbb{Z}^+\}$ and $\{\lfloor m\beta \rfloor : m \in \mathbb{Z}^+\}$, where m is a positive integer, partition the positive integers into two disjoint sets.*

Since ϕ is irrational and $\frac{1}{\phi} + \frac{1}{\phi^2} = 1$, Beatty's theorem implies that the sequences $\{\lfloor m\phi \rfloor : m \in \mathbb{Z}^+\}$ and $\{\lfloor m\phi^2 \rfloor : m \in \mathbb{Z}^+\}$ partition the positive integers. Since both these sequences are strictly increasing, we deduce that

$$\mathsf{MEX}(\{\lfloor k\phi \rfloor : k < n\} \cup \{\lfloor k\phi^2 \rfloor : k < n\}) = \min\{\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor\};$$

this equals $\lfloor n\phi \rfloor$, as desired. □

To summarize, Theorem 2 characterizes each P-position (x_n, y_n) as the result of a systematic recursion process. This approach highlights the dynamic relationship between successive P-positions, as each subsequent position is determined by the Minimum Excluded Value (MEX) of the set of coordinates from preceding P-positions. Theorem 3 introduces a geometric interpretation, illustrating the spatial distribution of P-positions along two distinct lines: $y = \phi x$ and $y = \frac{1}{\phi}x$. This representation capitalizes on the inherent symmetry of Wythoff Nim, enabling a systematic enumeration of P-positions through predictable patterns of movement along the x-axis and y-axis.

Chapter 3

Generalizations of Wythoff Nim, and their Solution

In this chapter we define some generalizations of Wythoff's game, and show how to extend the method of the previous chapter to solve some of these variants.

In the previous chapter we discussed taking non-negative integer multiples of the vector (ϕ, ϕ^2) and then using the floor function to convert it to an integer. When combined with the similarly treated integer multiples of (ϕ^2, ϕ), this is precisely the P-positions of Wythoff Nim. We call this method the 'stepping method' due to its reliance on the step sizes between consecutive P-positions. Below we will be using different step vectors to solve variants of the game.

Before jumping into the variants of the game, we will analyze the original Wythoff game using a different way of deriving the step vector. We will then apply these ideas to the generalized forms of the game.

3.1 Analyzing Wythoff's Game using Densities and Velocity

When analyzing the P-positions graph in the Wythoff game, we uncover patterns extending beyond the golden ratio slope. Notably, each row, column, and diagonal hosts precisely one P-position. This phenomenon arises because once a P-position is present in a column, it cannot lead to another P-position by descending down that same column. This logic extends to rows and diagonals. Hence, our strategy involves systematically traversing column by column to identify all P-positions. In each column, we ascend until encountering a position devoid of another P-position to the left and without any P-position sharing the same diagonal. Eventually, we locate a position exclusively leading to N-positions in that column, given the finite number of P-positions preceding it.

Another noteworthy observation pertains to the step sizes between consecutive P-positions. To transition from one P-position to the next along the upper line, we either take one step right and two steps up, or two steps right and three steps up. Symmetrically, the bottom line involves either two steps right and one step up, or one step right and two steps up. We are particularly interested in determining the average step size in both directions.

To explore the average step size and its relationship with the slopes of the lines in the Wythoff game, we introduce the concepts of velocity and density. If two piles have equal sizes, the position cannot be a P-position. We designate the upper line as the line where the size of the first pile is less than the size of the second pile, and the lower line as the line where the size of the second pile is less than the size of the first pile. These lines are reflections of each other due to the symmetry of P-positions.

Definition 3 (Velocity)**.** *The velocity of a line represents the average difference between two consecutive P-positions in the x-direction.*

Through a recursive process, we identify infinitely many P-positions. By sorting the upper line based on the x-coordinates (ensuring no two positions have the same x-coordinate), we examine the first n points and calculate the average difference between consecutive points. Taking n to infinity provides the velocity of the upper line.

In the context of the Wythoff game, the velocity of the upper line falls between 1 and 2, as transitioning from one P-position to the next involves either a one-step or two-step movement to the right. Similarly, the velocity of the lower line ranges between 2 and 3. Alternatively, if we denote the nth P-position on the upper line as (x_n, y_n), the velocity of the upper line can also be defined as $\lim_{n\to\infty} \frac{x_n}{n}$. Given the symmetry of P-positions, the nth P-position on the lower line has its x-coordinate equal to y_n, leading to the definition of the lower line's velocity as $\lim_{n\to\infty} \frac{y_n}{n} = \lim_{n\to\infty} \frac{x_n}{n} + 1$.

Definition 4 (Density)**.** *The density of a line is defined as the reciprocal of its velocity, representing the number of P-positions on the line within one unit in the x-direction.*

Density provides insight into the spatial distribution of P-positions along a line, offering a complementary perspective to velocity. It can be interpreted as the average number of P-positions within one unit interval along the x-direction. Additionally, density can be viewed as the probability of a P-position occurring at a specific x-coordinate.

Given that each column in the Wythoff game contains exactly one P-position, the total density of both lines equals 1. Let Δx denote the velocity of the upper line. We can express this relationship using the equation:

$$\frac{1}{\Delta x} + \frac{1}{\Delta x + 1} = 1.$$

Solving this equation yields $\Delta x = \phi$, where ϕ represents the golden ratio. Consequently, the average step size between P-positions on the upper line in the x-direction is ϕ, while in the y-direction, it is $\phi + 1$. This result allows us to determine the slope of the upper line by dividing the two step sizes: $\frac{\phi+1}{\phi} = \phi$. This aligns with our previous findings from the preceding chapter.

As we've established, the P-positions of Wythoff Nim lie on the lines $y = \phi x$ and $y = \frac{1}{\phi}x$. Now, let's visualize these lines on the graph. Upon observation, we notice a distinctive pattern: each P-position on the upper line intersects the top and bottom sides of the square but never crosses the left or right sides. Conversely, P-positions on the lower line intersect the left and right sides of the square while avoiding the top or bottom sides. This property will be formally proven in the following section, further elucidating the geometric characteristics of Wythoff Nim's P-positions. We refer to this characterization of P-positions as the line method.

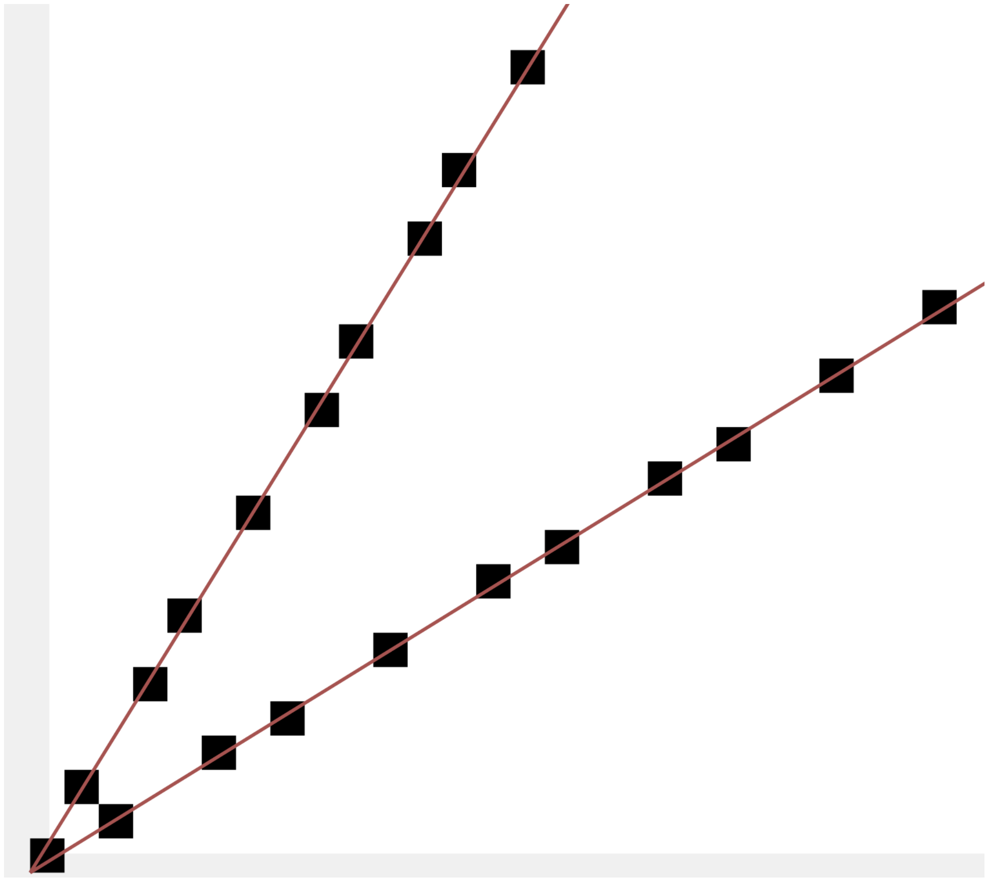

Figure 3.1: The lines pass through opposite sides of the squares.

3.2 The Game $W(a,a)$: Adding the Multiplicative Rules

In this chapter, following Larsson [5], we will define games using rule sets, which consist of tuples of natural numbers. Players can select a tuple on their turn and subtract a multiple of this tuple from the piles. For instance, the rule set for Wythoff Nim is defined as

$$(0,1),(1,0),(1,1).$$

In the following sections, we delve into the modification of the Wythoff Nim game $W(a,a)$, governed by the rules

$$(a,0),(0,a),(1,1).$$

These rules allow players to either take a multiple of a from the first pile, a multiple of a from the second pile, or an equal number from both piles during each move. This scenario represents a specific case of a more generalized rule set

$$(a,0),(0,b),(1,1),$$

where players can extract multiples of a and b from the respective piles, or an equal number from both piles at each turn. We will explore the general case further in the subsequent section.

Previously, we defined three mathematical methods to represent the set of squares in the plane corresponding to P-positions. Here, we examine how these methods extend to the game $W(a,a)$ and assert their equivalence for this specific scenario.

1. The Line Method: In this approach, we define two lines originating from $(0,0)$, each with a unique slope determined by the values of a and b. The set of P-positions comprises the square cells intersected by this line as it traverses opposite sides of the squares. For the game $W(a,a)$, the P-positions manifest as two diagonal lines, with the upper line intersecting the top and bottom of the squares, and the lower line intersecting the left and right sides.
2. The stepping method. Here, we initiate a traversal from $(0,0)$, progressing in steps of size $(\Delta x, \Delta y)$. Each square encountered during this traversal constitutes a P-position on the upper line. The P-positions on the lower line are derived similarly, but with step sizes $(\Delta y, \Delta x)$ due to symmetry. Specifically, the P-positions are located at coordinates $(\lfloor n\Delta x \rfloor, \lfloor n\Delta x \rfloor + n)$ and $(\lfloor n\Delta x \rfloor + n, \lfloor n\Delta x \rfloor)$, where $n = 0, 1, 2, ...$ and Δx denotes the velocity vector in the x-direction.
3. This method follows the standard recursive approach for determining P-positions, wherein:
   - All terminal positions, such as (0, 0) in this game, are considered P-positions.
   - Positions where all possible moves lead to N-positions are classified as P-positions.
   - Positions where at least one move results in a P-position are categorized as N-positions.

To establish the equivalence of the three definitions, it suffices to demonstrate that the line method aligns with both the stepping method and the recursive definition.

3.2.1 Slope Calculation for $W(a,a)$

Following our analysis of the Wythoff Nim game $W(a,a)$ in the previous section, where $a = 1$, let's now examine the scenario for $a > 1$ under the rules $(a,0), (0,a), (1,1)$. In this variant, the P-positions form two lines that exhibit symmetry with respect to the diagonal $y = x$.

Consider the average step size between two adjacent P-positions on the upper diagonal as $(\Delta x, \Delta y)$. By symmetry, the average step size between two adjacent P-positions on the lower diagonal becomes $(\Delta y, \Delta x)$.

It's noteworthy that each row and column consists of exactly a P-positions with distinct modular a values. This occurs because moving from one P-position to another doesn't involve taking a multiple of a from the first pile. We can utilize an argument analogous to the one employed in analyzing Wythoff Nim: filling the P-positions column by column, we ascend within each column until encountering a P-position that does not lead to another P-position that is multiple of a away on the left or on the same 45-degree diagonal. Moreover, it cannot intersect the P-positions below in the same column by taking a multiple of a. Thus, we precisely identify a P-positions in each column.

Similarly, each diagonal hosts precisely one P-position, maintaining the unmodified rule $(1,1)$ from standard Wythoff Nim. Consider the intersection of the diagonal $y = x + 1$ and the upper line; we obtain the vector $(\Delta x, \Delta y)$. Since the diagonal $y = x+1$ intersects the y-axis at $(0,1)$, we can interpret this as projecting the vector $(\Delta x, \Delta y)$ along the diagonal onto the y-

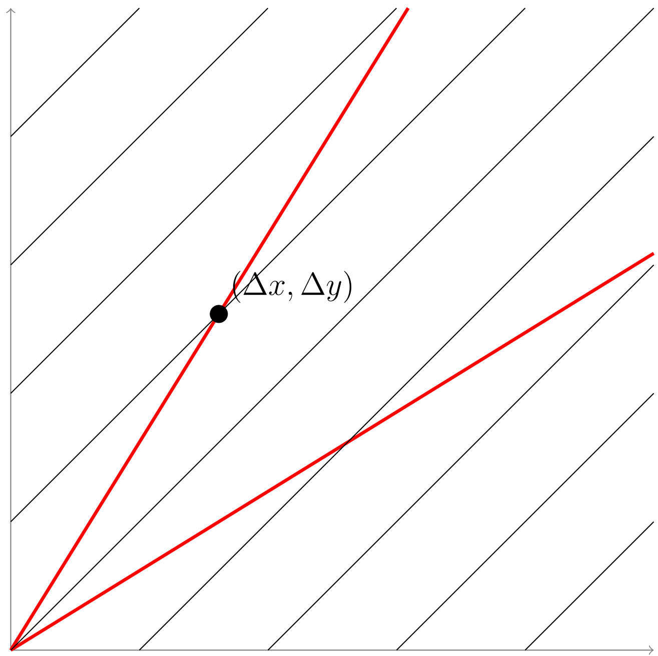

Figure 3.2: the upper line of P-positions intersecting $y = x + 1$

axis. In other words, Δx is projected to become 0, and Δy is projected to become $\Delta y - \Delta x = 1$. We can apply the same process for the diagonal $y = x + 2$, resulting in $(2\Delta x, 2\Delta y)$, and so on. Since each $y = x + c$ diagonal can intersect only one P-position, each $(0, c)$ is intersected exactly once. Consequently, the fraction of integer points intersected on the y-axis is 1, leading to the equation $\frac{1}{\Delta y - \Delta x} = 1$ [4]. With these insights, we derive the following set of equations, applying the same reasoning as in the previous section:

$$\frac{1}{\Delta x} + \frac{1}{\Delta y} = a$$
$$\frac{1}{\Delta y - \Delta x} = 1$$

Solving these equations yields:

$$\Delta x = \frac{2 - a + \sqrt{a^2 + 4}}{2a}$$
$$\Delta y = \frac{2 + a + \sqrt{a^2 + 4}}{2a}$$

Subsequently, the slope of the upper diagonal is given by $m = \dfrac{\Delta y}{\Delta x} = \dfrac{2 + a + \sqrt{a^2 + 4}}{2 - a + \sqrt{a^2 + 4}}$, while the slope of the lower diagonal is $\frac{1}{m} = \frac{\Delta x}{\Delta y}$. Additionally, it's worth noting that $\Delta y = \Delta x + 1$ and the difference between the two slopes is exactly a.

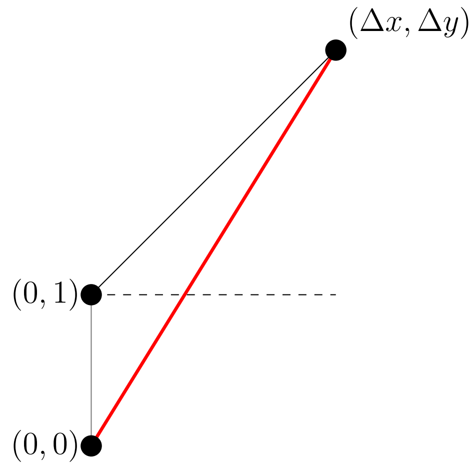

Figure 3.3: Projecting $(\Delta x, \Delta y)$ onto the y-axis

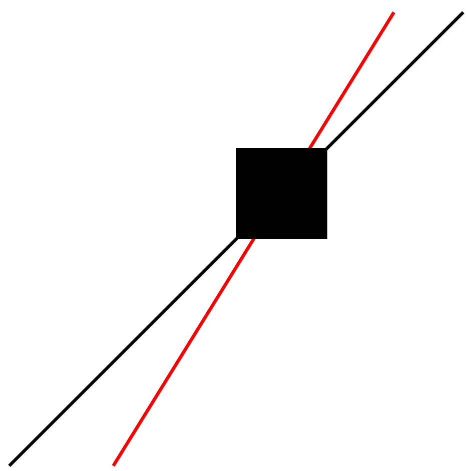

Figure 3.4: The intersection of the upper line and a 45 degree diagonal

3.2.2 The Line Method ⇔ The Stepping Method

Theorem 5. *In $W(a, a)$, the P-positions formed by the upper diagonal are represented by the coordinates ($\lfloor n\Delta x \rfloor, \lfloor n\Delta x \rfloor + n$), while the P-positions formed by the lower diagonal have symmetric representations as ($\lfloor n\Delta x \rfloor + n, \lfloor n\Delta x \rfloor$).*

Proof. First, note that the P-positions represented by the line method precisely correspond to the squares that result from the intersections of the derived upper and lower diagonals and the $y = x + n$ lines for $n \in \mathbb{Z}$. To illustrate this, let's consider the upper diagonal in the line method. Suppose the diagonal intersects a $y = x + n$ line within a square. Given that the slope of the upper diagonal exceeds 1 and the $y = x + n$ line passes through both corners of the square, the upper diagonal must traverse the top and bottom sides of the square. Conversely, if a square is the intersection of the upper diagonal and one of the $y = x + n$ lines, it qualifies as a P-position according to the definition in the line method, for the same rationale. A similar argument applies to the lower diagonal, demonstrating that it traverses the left and right sides of the square when intersecting a $y = x + n$ line.

To demonstrate the equivalence of this representation with the stepping method, let's consider an arbitrary $y = x + n$ line. The upper diagonal, derived from the slope calculations, corresponds to the line $y = \frac{\Delta y}{\Delta x} x$. By setting these two equations equal, we can find the intersection (x', y') of the line with the upper diagonal. Equating $y = \frac{\Delta y}{\Delta x} x$, which simplifies to $y = (\frac{1}{\Delta x} + 1)x$, with $y = x + n$, yields $x' = n \cdot \Delta x$ and $y = x' + n$ upon solving. Taking the floors of these values results in the closed form of P-positions as desired.

□

3.2.3 The Line Method ⇔ The Recursive Definition of P-Positions

Theorem 6. *In the game $W(a, a)$, upon plotting the upper and lower diagonals containing the P-positions on a grid, the P-positions precisely correspond to the squares intersected entirely by these lines. Specifically, P-positions are the squares where the upper diagonal traverses the top and bottom sides, and the lower diagonal traverses the left and right sides.*

Illustrated in Figure 3.5 for the case when $a = 2$, observe that the number of consecutive P-positions is always either 1 or 2 on both the upper and lower lines. This means there are either 1 or 2 consecutive P-positions in the same column on the upper line and similarly either 1 or 2 consecutive P-positions in the same row on the lower line.

Proof. We define squares not entirely traversed by the diagonals as white squares, while squares entirely traversed by both diagonals are black squares. Our goal is to demonstrate that white squares represent N-positions and black squares represent P-positions. It suffices to prove two key points:

1. For every white square, there exists a move leading to a black square.
2. All moves from a black square only lead to white squares.

Recall the calculated slopes of the diagonals: $m = 1 + \dfrac{2a}{2 - a + \sqrt{a^2 + 4}}$ and $\frac{1}{m} = m - a$. Notably, $a < m < a + 1$ always. Observing $a - 1$ or a consecutive black squares on both upper and lower lines, we note that there can't be more than a consecutive black squares on the upper line ($m < a + 1$) or fewer than $a - 1$ ($m > a$). Otherwise, the slope of the upper line would be greater than $a - 1$ or less than a. Thus, the upper line traverses either a or $a - 1$ black squares in each column, with similar behavior for the lower line in rows.

In each column, if the upper line intersects $a - 1$ black squares, the lower line intersects one black square. Moreover, the black squares intersected by the lower line must have a different modular a value than those intersected by the upper line in that column. Conversely, if the upper line intersects a black squares, then the lower line does not cross any black squares in that column. This point will be elucidated when we prove the second key point.

Similarly, each row intersects the upper line at exactly one black square if the lower line intersects $a - 1$ P-positions in that row, and the a black squares in that row all have different modular a values. Conversely, the upper line does not intersect any black squares in the row if the lower line intersects a P-positions in that row.

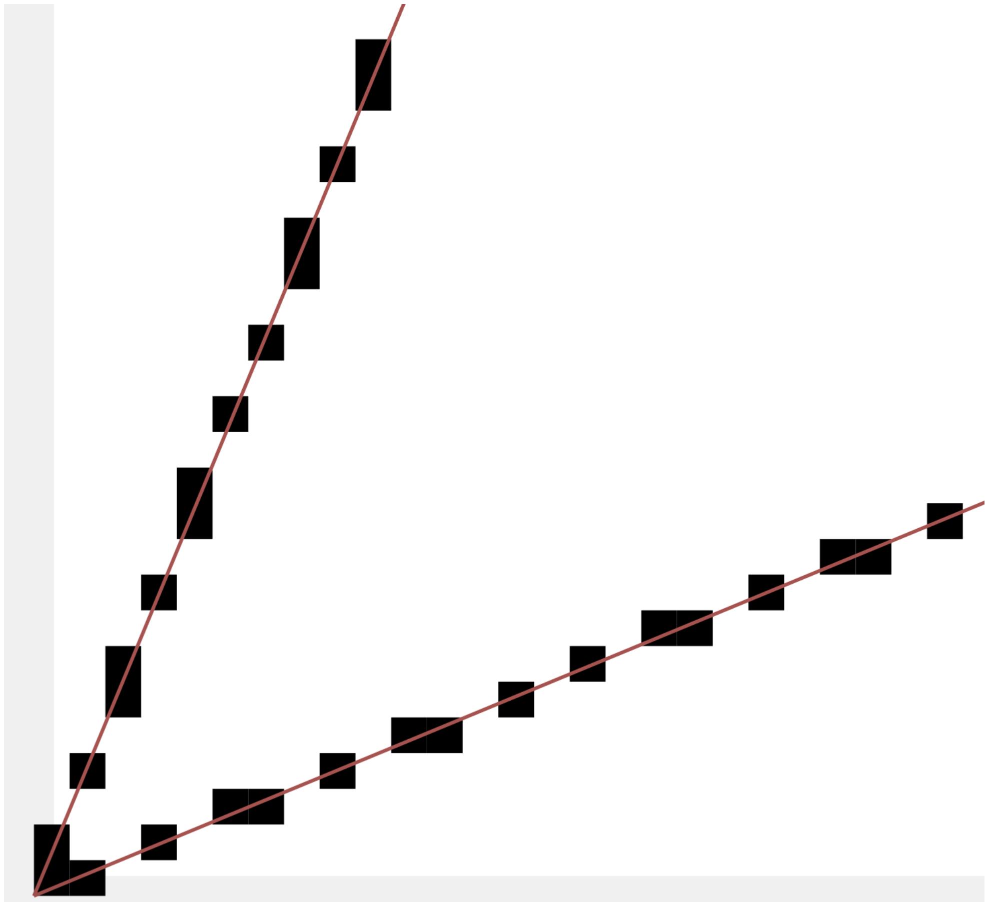

Figure 3.5: $W(a, a)$ when $a = 2$

Additionally, considering the $y = x + c$ diagonals, where c is an integer, each diagonal intersects exactly one black square. Due to the upper line's slope being above 1 and the lower line's slope below 1, each $y = x + c$ diagonal intersects a P-position on one of the two lines.

To prove the first key point, we aim to demonstrate that for every white square, there exists a move leading to a black square.

Case 1: If the white square lies to the left of the upper diagonal. Taking the move $(0, a)$ allows us to descend the column until encountering a black square. As all black squares in this column possess different modular a values, encountering a black square is ensured.

Case 2: If the white square lies to the right of the lower diagonal. By symmetry to Case 1, the move $(a, 0)$ enables us to move to the left along the row until encountering a black square.

Case 3: If the white square lies between the lower and upper diagonals. In this scenario, taking the move $(1, 1)$ allows movement along the 45-degree diagonal. Given that each $y = x + c$ line intersects precisely one P-position, encountering a black square along this line is inevitable.

Now, let's move on to part 2 of the proof.

We establish that each black square does not lead to any other black squares. Initially, we consider only the upper diagonal, with similar reasoning applying to the lower diagonal due to symmetry. Three types of moves can be made from a given square:

1. Left: As no black squares exist to the left of the upper diagonals, left moves cannot lead to another black square.
2. Down along the 45-degree diagonal: Since each $y = x + c$ line intersects only one black square, moving down the diagonal from a black square does not lead to another black square.
3. Down: We analyze this move in two cases:
   (a) If the upper diagonal crosses a consecutive black squares in this column. We aim to demonstrate that the lower diagonal does not intersect any black squares in that column. Consider the square where the lower diagonal intersects its left side. We want to show that the lower diagonal goes through its top side instead of its right side, which makes it a white square. In this scenario, the a squares on the upper diagonal have different modular a values since they are consecutive. Notably, the slope difference between the two diagonals is a. Let x_l be the x-coordinate of the left side of the column, and $x_l + 1$ be the x-coordinate of the right side of the column. The gap between the two diagonals on the left side is ax_l, while on the right side, it is $a(x_l + 1) = ax_l + a$. Now, let's draw another line l with the same slope as the lower diagonal that intersects the upper diagonal at x_l. Then, the y-intersection of l with $x_l + 1$ is a units down from the y-intersection of the upper line with $x_l + 1$, so l crosses the bottom side of the bottom-most black square in that column, hence it does not go through the left and right sides of a square. Since l is parallel to the lower line, translating l by ax_l units down gives us the lower line. Suppose the lower diagonal intersects the left side of a square in that column, it cannot intersect the right side of that square because it should exhibit the same behavior as l. Otherwise, the gap on the right side cannot remain $ax_l + a$. Thus, the lower diagonal can only intersect a white square in that column.

   (b) If the upper diagonal crosses only $a - 1$ consecutive black squares. Again, we define x_l to be the x-coordinate of the left side of the column, and we draw another line l with the same slope as the lower diagonal that intersects the upper diagonal at x_l. Same as the previous case, the gap between the upper diagonal and l at $x_l + 1$ is a. However, since the upper diagonal only crosses $a-1$ black squares here, l crosses the right side of the white square right beneath the bottom-most black square to maintain this gap. This white square has a different modular a value than the $a - 1$ black squares on the upper diagonal since it forms a consecutive squares with the black squares. Now we translate l downwards by ax_l to obtain the lower line. Once again, the lower line follows the same behavior and crosses the left and right sides of a square, making this a black square. Furthermore, since we translated exactly ax_l units down, it has the same modular a value as the white square that is crossed by l, meaning the black square on the bottom diagonal has a different modular a value than all other black squares in this column.

□

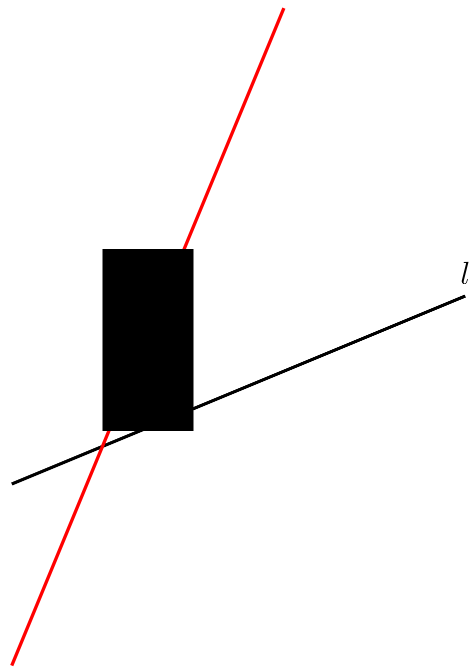

Figure 3.6: The upper diagonal crosses a consecutive black squares, $a = 2$

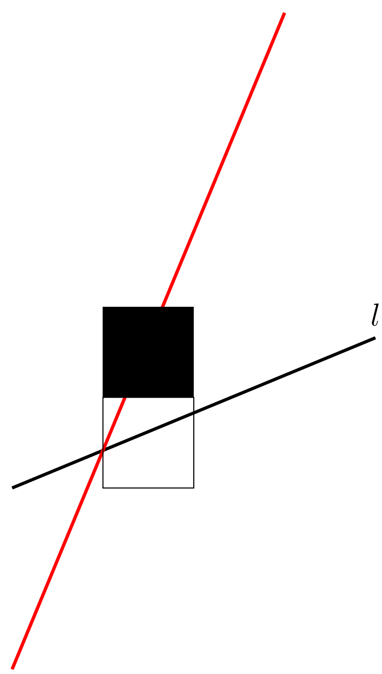

Figure 3.7: The upper diagonal crosses $a - 1$ consecutive black squares, $a = 2$

3.3 The Game $W(a,b)$: a More Generalized Case

We can extend the game $W(a,a)$ to $W(a,b)$, where the rule set becomes $(a,0),(0,b),(1,1)$. This means that in each turn, players can take a multiple of a from the first pile, or a multiple of b from the second pile, or an equal number from both piles. Later, we will observe that the solution vectors are no longer symmetric, but a generalizable pattern emerges for $W(a,b)$.

Let Δx_1 and Δy_1 represent the velocity vectors in the x and y directions respectively for the upper diagonal. Similarly, let Δx_2 and Δy_2 denote the velocity vectors in the x and y directions respectively for the lower diagonal. We claim that the solution vectors for the upper diagonal are ($\lfloor n\Delta x_1 \rfloor$, $\lfloor n\Delta x_1 \rfloor + n$), and for the lower diagonal, they are ($\lfloor n\Delta y_1 \rfloor + n$, $\lfloor n\Delta y_1 \rfloor$) based on the stepping method.

3.3.1 Slope Calculation for $W(a,b)$

In $W(a,b)$, the P-positions of this game form two lines that are no longer symmetric due to the different rules for each pile.

Now, let's consider the average step size between two adjacent P-positions on the upper diagonal to be (Δx_1, Δy_1), and the average step size between two adjacent P-positions on the lower diagonal to be (Δx_2, Δy_2).

Using analogous reasoning to $W(a,a)$, we observe that each row contains exactly a P-positions, each column contains exactly b P-positions, and each diagonal has exactly one P-position. With this information, we can derive the following set of equations:

$$\frac{1}{\Delta x_1} + \frac{1}{\Delta x_2} = b$$
$$\frac{1}{\Delta y_1} + \frac{1}{\Delta y_2} = a$$
$$\frac{1}{\Delta y_1 - \Delta x_1} = 1$$
$$\frac{1}{\Delta x_2 - \Delta y_2} = 1$$

which solves to

$$\Delta x_1 = \frac{2a - ab + \sqrt{b^2a^2 + 4ab}}{2(ab - a + b)}.$$

$$\Delta y_1 = \Delta x_1 + 1 = \frac{2b + ab + \sqrt{b^2a^2 + 4ab}}{2(ab - a + b)}.$$

and

$$\Delta x_2 = \frac{2a - ab + \sqrt{b^2a^2 + 4ab}}{2a - 2b - b^2a + b \cdot \sqrt{b^2a^2 + 4ab}}.$$

$$\Delta y_2 = \Delta x_2 - 1 = \frac{2b - ab + b^2a + (1-b)\sqrt{b^2a^2 + 4ab}}{2a - 2b - b^2a + b \cdot \sqrt{b^2a^2 + 4ab}}.$$

The slope of the upper diagonal is calculated as

$$m = \frac{\Delta y_1}{\Delta x_1} = \frac{2b + ab + \sqrt{b^2a^2 + 4ab}}{2a - ab + \sqrt{b^2a^2 + 4ab}} = \frac{ab + \sqrt{b^2a^2 + 4ab}}{2a},$$

while the slope of the lower diagonal is $\frac{\Delta y_2}{\Delta x_2} = m - b$. It's crucial to note that the difference between the two slopes is exactly b. Similar to the game $W(a, a)$, the slope of the upper diagonal exceeds 1, and the slope of the lower diagonal is less than 1 but greater than 0.

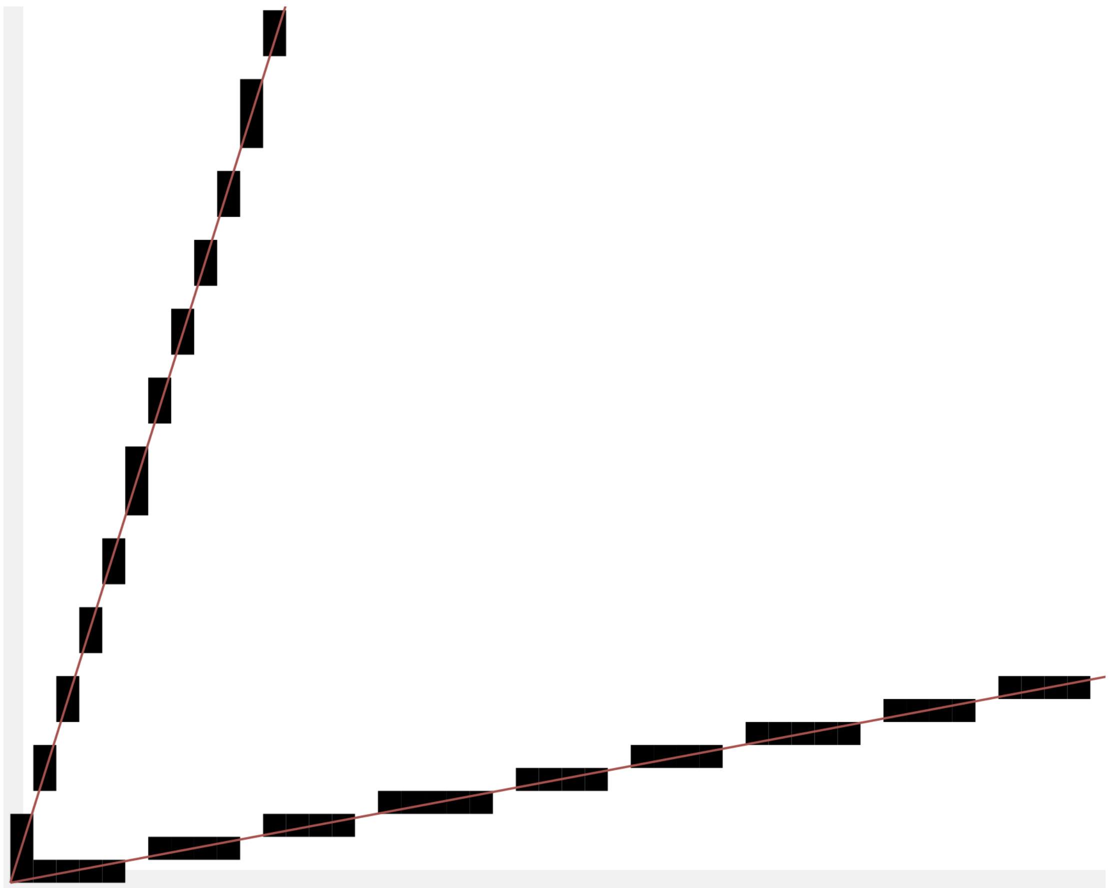

Figure 3.8: $W(a, b)$ when $a = 5$ and $b = 3$

3.3.2 The Line Method ⇔ The Stepping Method

Theorem 7. *In $W(a, b)$, the P-positions formed by the upper diagonal are represented by the coordinates $(\lfloor n\Delta x_1 \rfloor, \lfloor n\Delta x_1 \rfloor + n)$, while the P-positions formed by the lower diagonal are represented by the coordinates $(\lfloor n\Delta x_2 \rfloor, \lfloor n\Delta x_2 \rfloor - n)$.*

Proof. Similar to the $W(a, a)$ game, the P-positions represented by the line method precisely correspond to the squares resulting from the intersections of the derived upper and lower diagonals and the $y = x + n$ lines for $n \in \mathbb{Z}$. The proof follows from the same reasoning as in

Theorem 5, as we again have the slope of the upper diagonal greater than 1 and the slope of the lower diagonal between 0 and 1.

To demonstrate the equivalence of this representation with the stepping method, let's consider the P-positions on the upper and lower diagonals separately:

Firstly, let's consider an arbitrary $y = x + n$ diagonal for $n > 0$. This line intersects with the upper diagonal. The upper diagonal, derived from the slope calculations, corresponds to the line $y = \frac{\Delta y_1}{\Delta x_1}x$. By setting these two equations equal, we can find the intersection (x', y') of the line with the upper diagonal. Equating $y = \frac{\Delta y_1}{\Delta x_1}x$, which simplifies to $y = (\frac{1}{\Delta x_1} + 1)x$, with $y = x + n$, yields $x' = n \cdot \Delta x_1$ and $y' = x' + n$ upon solving. Taking the floors of these values results in the closed form of P-positions as desired.

Secondly, let's examine an arbitrary $y = x - n$ diagonal for $n > 0$. This line intersects with the lower diagonal. The lower diagonal, derived from the slope calculations, corresponds to the line $y = \frac{\Delta y_2}{\Delta x_2}x$. By setting these two equations equal, we determine the intersection (x', y') of the line with the lower diagonal. Equating $y = \frac{\Delta y_2}{\Delta x_2}x$, which simplifies to $y = (1 - \frac{1}{\Delta x_2})x$, with $y = x - n$, gives $x' = n \cdot \Delta x_2$ and $y' = x' - n$ upon solving. The floors of these values yield the closed form of P-positions as intended. □

3.3.3 The Line Method ⇔ The Recursive Definition

Theorem 8. *In the game $W(a, b)$, the P-positions align precisely with the squares intersected entirely by the upper and lower diagonals on a grid. Specifically, P-positions are where the upper diagonal traverses the top and bottom sides, and the lower diagonal traverses the left and right sides.*

Illustrated in Figure 3.8 for the case when $a = 5$ and $b = 3$, observe that the number of consecutive P-positions is always either 2 or 3 on both the upper line, and the number of consecutive P-positions is always either 4 or 5 on both the lower line.

Proof. We'll define black and white squares, similar to Theorem 6. The proof centers on two key points:

1. For every white square, there exists a move leading to a black square.
2. All moves from a black square only lead to white squares.

Let's revisit the calculated slope of the upper diagonal: $m = \dfrac{ab + \sqrt{b^2a^2 + 4ab}}{2a}$. As in the $W(a, a)$ game, $b < m < b + 1$ always. Similarly, on the upper line, we observe either $b - 1$ or b consecutive black squares. Considering the game $W(b, a)$, the lower diagonal in $W(a, b)$ mirrors the upper diagonal in $W(b, a)$ across the $y = x$ line. Consequently, if the upper line in $W(b, a)$ features $a - 1$ or a consecutive black squares, the lower line in $W(a, b)$ will exhibit the same pattern.

Within each column, when the upper line intersects $b - 1$ black squares, the lower line intersects one black square. Furthermore, the black squares intersected by the lower line must have a different modular b value than those intersected by the upper line in that column. Conversely, if the upper line intersects b black squares, then the lower line does not intersect any black squares in that column.

However, in $W(a,b)$, the symmetry seen in $W(a,a)$ is disrupted. If the lower line intersects $a-1$ black squares in a row, the upper line intersects one black square in that row. Conversely, if the lower line intersects a black squares, then the upper line doesn't intersect any black squares in that row.

Just as in $W(a,a)$, each $y = x + c$ diagonal intersects exactly one black square. With the upper line's slope above 1 and the lower line's slope below 1, each $y = x + c$ diagonal intersects a P-position on one of the two lines.

To establish the first key point, we aim to show that for every white square, there exists a move leading to a black square.

Case 1: If the white square is positioned to the left of the upper diagonal. We take the move $(0, b)$ to descend the column until reaching a black square. As each black square in this column has a distinct modular b value, encountering a black square is assured.

Case 2: If the white square is located to the right of the lower diagonal. We employ the move $(a, 0)$ to move left along the row until encountering a black square.

Case 3: If the white square lies between the lower and upper diagonals. In this scenario, we utilize the move $(1, 1)$ to travel along the 45-degree diagonal. Since each $y = x+c$ line intersects precisely one P-position, encountering a black square along this line is inevitable.

Now, let's proceed to the second part of the proof.

We establish that each black square does not lead to any other black squares. First, we consider the black squares on the upper diagonal, applying reasoning akin to that in part 2 of the proof in Theorem 6. Then, we examine the upper diagonal in the game $W(b,a)$, employing the symmetric argument for the lower diagonal in the game $W(a,b)$.

1. Left or Down along the 45-degree diagonal: We employ the same argument as in part 2 of the proof in Theorem 6.
2. Down: We analyze this move in two cases, using a similar argument as in Theorem 6:
   (a) If the upper diagonal crosses b consecutive black squares in this column: Our objective is to demonstrate that the lower diagonal does not intersect any black squares in that column. Given the slope difference between the two diagonals is b, we define l as before and apply the identical argument from the previous proof to establish that the lower diagonal can only intersect a white square in that column.
   (b) If the upper diagonal crosses only $b-1$ consecutive black squares in this column: Again, we utilize the same proof to demonstrate that the lower diagonal crosses one black square in that column, with that black square having a different modular b value than all other black squares in this column.

Now consider a black square on the upper diagonal of the game $W(b,a)$, then project each move to the corresponding black square on the lower diagonal of the game $W(a,b)$.

1. Left in $W(b,a)$ (or down in $W(a,b)$) or Down along the 45-degree diagonal: we use the identical argument as in the part 2 of the proof in Theorem 6. Moving left from a black square on the upper diagonal of $W(b,a)$ is symmetric to moving down from the black square on the lower diagonal of $W(a,b)$. Moving down a 45-degree diagonal is identical in both games.

2. Down in $W(b, a)$ (or left in $W(a, b)$): We analyze this move in two cases, employing a similar argument as in Theorem 6:
   (a) If the upper diagonal of $W(b, a)$ crosses a consecutive black squares in this column: Our goal is to demonstrate that the lower diagonal does not intersect any black squares in that column. Given the slope difference between the two diagonals is a, we define l as before and use the identical argument from the previous proof to establish that the lower diagonal can only intersect a white square in that column.
   (b) If the upper diagonal of $W(b, a)$ crosses only $a - 1$ consecutive black squares in this column: Again, we utilize the same proof to show that the lower diagonal intersects one black square in that column, with that black square having a different modular a value than all other black squares in this column.

   We have demonstrated that from any black square on the lower diagonal of $W(a, b)$, it cannot lead to another black square by taking any allowed move. We established this by showing that from any black square on the upper diagonal of $W(b, a)$, it also cannot lead to another black square by taking any allowed move.

□

Chapter 4

Fixing the Starting State of P-positions

We introduce another variant of Wythoff Nim where we fix a random fraction of a $k \times k$ box ($k > 0$) as initial P-positions and examine how this affects the patterns of P-positions in the long run. For instance, if we choose a fraction of 1, then all positions within the box from $(0, 0)$ to $(k-1, k-1)$ are predetermined as P-positions. Conversely, if we choose a fraction of 0, then none of the positions within this range are considered P-positions. Otherwise, we randomly select this fraction of positions within the box to be our initial P-positions. Our simulations suggest that depending on the box size and the chosen fraction, the P-positions appear to converge to two lines with slopes identical to those in regular Wythoff Nim, though the patterns may not entirely overlap.

An intriguing observation arises when considering the scenario where all positions within a 4×4 box are defined as P-positions, as depicted in Figure 4.1. Here, the P-positions appear to align along lines with slopes identical to those observed in regular Wythoff Nim. However, these lines originate from $(-3, -3)$ instead of the origin. This behavior persists even when considering larger box sizes; for instance, in the case of a 8×8 box, depicted in Figure 4.6, although the P-positions initially exhibit slight fluctuations, they eventually appear to converge to two lines with slopes corresponding to those in standard Wythoff Nim. Notably, these lines originate from $(-7, -7)$. To formalize and generalize this behavior, we propose the following conjecture:

Conjecture 1. *If we designate all initial positions within a $k \times k$ grid as P-positions, after a finite number of iterations, the P-positions will converge to two beams with slopes ϕ and $\frac{1}{\phi}$, originating from $(-k+1, -k+1)$.*

Now, let us delve into a novel approach for generating P-positions in this variant of Wythoff Nim.

4.1 Expanding the Diagonals

In the previous chapter, we discussed filling P-positions by columns, where we identify the first available position in each column that doesn't intersect any P-position to the left or on the same diagonal. In this chapter, we introduce a new method of filling P-positions column by column by populating the unfilled diagonals, represented as $y = x+c$ for $c \in \mathbb{Z}$, from the center outward. As

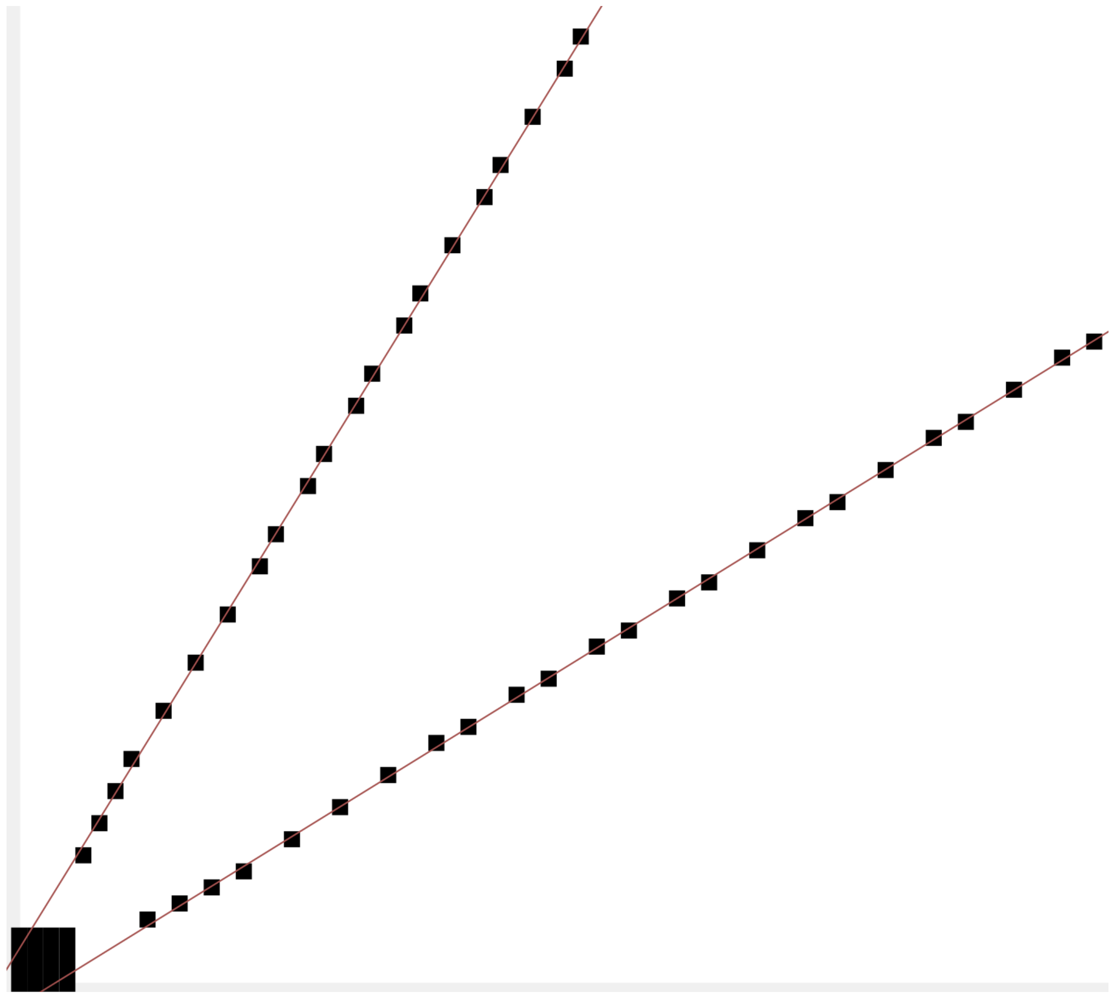

Figure 4.1: Initial 4×4 positions predefined as P-positions

we progress in filling out the diagonals, the patterns gradually converge to those of P-positions in Wythoff Nim. Eventually, we reach a stage where a contiguous set of diagonals are fully filled, and thereafter, we expand the filled diagonals outward to observe whether the resulting patterns align with those of Wythoff Nim. To demonstrate that we can generate all P-positions using this method, we establish two theorems that outline the process of generating P-positions through induction. We define a diagonal, column, or row as "filled" or "occupied" if it contains at least one P-position.

Theorem 9. *Suppose we designate all positions in the initial $k \times k$ box as P-positions. We can then systematically fill all P-positions column by column by expanding diagonals from $y = x$. Let t represent the current column or time, $u = x + c_u$ denote the topmost filled diagonal, and $l = x + c_l$ denote the lowest filled diagonal. Furthermore, let l_y represent the y-coordinate of the intersection of l with column t, and u_y represent the y-coordinate of the intersection of u with column t. As we progress, the following conditions consistently hold:*

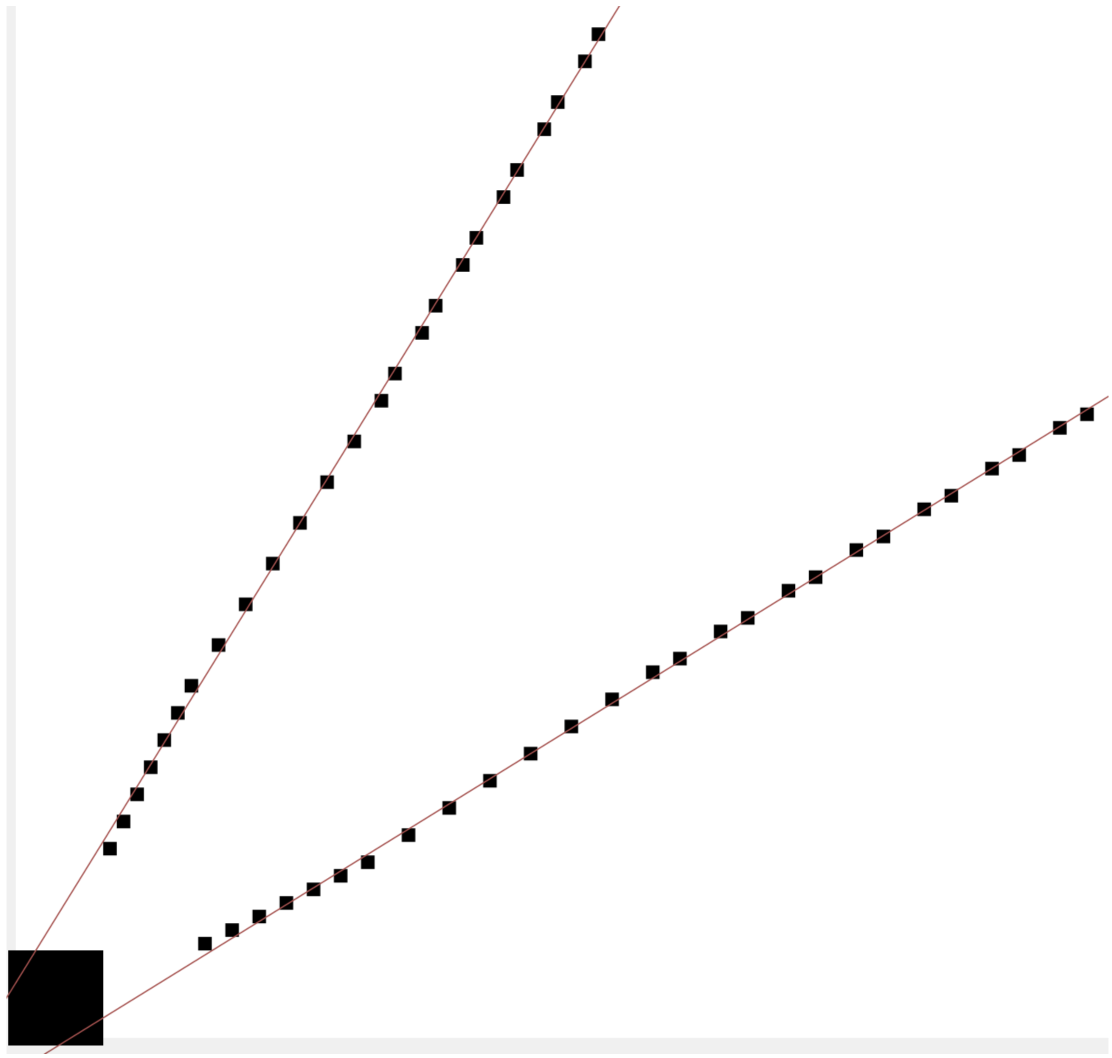

Figure 4.2: Initial 8×8 positions predefined as P-positions

1. *All diagonals between l and u (inclusive) are filled, meaning $y = x + c$ contains a P-position for all $c_u \leq c \leq c_l$.*
2. *All rows below $l_y - 1$ are filled.*
3. *None of the rows above u_y are filled.*

Proof. Note that when we fill the next P-position in the next column, either l expands (or l becomes the diagonal right below l), or u expands (or u becomes the diagonal right above u). Similar to the Wythoff game, all rows and columns starting from a have exactly one P-position on them. Similarly, all diagonals that do not intersect the predefined P-positions have exactly one P-position on them.

Proceed by induction on t.

Base Case: Since all the columns up until $k - 1$ are filled, we are about to fill the column $t = k$. The next P-position that we fill in column k is at $2k$ so that u does not intersect the

P-position at $(0, k-1)$. Then $l = x - (k-1)$ and $u = x + k$. Since all the diagonals between l and u were filled with the predefined P-positions in the box, the first condition is satisfied. Since $l_y = 1$ and $k \geq 1$, all rows below $l_y - 1$ are filled. None of the rows above u_y is filled, so the theorem holds for the base case.

Induction Hypothesis: Each time we move to the next column, either l expands or u expands, and the new P-position is on either l or u. Assume the theorem holds for all columns $\leq t$.

Induction Step: Since there is only one P-position at time t, we consider whether the P-position at t is on l or u. Let l_y be the y-coordinate of l at time t, and let u_y be the y-coordinate of u at time t.

Case 1: At time t, the P-position is on l. Consider u' and l', the topmost and lowest diagonals respectively after we fill the P-position at $t+1$. At time $t+1$, we want to see whether the new P-position is on the diagonal expanded by l or the diagonal expanded by u. By the induction hypothesis, all rows up to $l_y - 1$ are filled, and the P-position on l_y is filled at time t. Thus, the P-position at $t+1$ must be above l_y. We look at all the positions at column $t+1$ above l_y. All the positions between l_y and $u_y + 1$ (u intersects $t+1$ at $u_y + 1$) are on a diagonal that's between l and u, and these diagonals have already been filled by the induction hypothesis. Thus, the next available position that is not on a filled diagonal or a filled row on column t is right above $u_y + 1$, which is $u_y + 2$, and it is on the diagonal right above u, which is u'. Thus, u expands to u' at $t+1$, and $u'_y = u_y + 2$. Since l did not expand, $l' = l$. Now we want to check if the theorem still holds at $t+1$. Now u' is the new u, and since u' was just filled and all diagonals between l and u are filled by the induction hypothesis, the first condition holds at $t+1$. Since l does not expand and the row l_y was filled at t, $l'_y - 1 = l_y$, so the second condition still holds. At $t+1$, we expanded u to u' and filled the row u'_y but none of the rows above it, thus the third condition still holds from the induction hypothesis.

Case 2: At time t, the P-position is on u. We further consider whether the row l_y has been filled previously.

- If row l_y is not filled, then we can expand l to l' and fill the P-position at $l'_y = l_y$ at time $t+1$. To check if the theorem holds in this subcase, we expanded l to l', so l' is filled, and all diagonals between l and u are filled by the induction hypothesis. Since $l'_y = l_y$, the second condition still holds by the induction hypothesis. We did not change u, so $u' = u$, and $u'_y = u_y + 1$. Then none of the rows above u'_y are filled by the induction hypothesis.
- If row l_y has been filled previously, then we arrive at a situation similar to Case 1. All the rows below and including l_y have been filled, and the next available position at $t+1$ is at $u_y + 2$. Then we expand u to u', and the new P-position is at $u'_y = u_y + 2$. The proof of correctness follows from Case 1.

Thus the theorem holds for both cases.

□

Now, consider a scenario where instead of predefining all positions in the $k \times k$ box as P-positions, a fraction of positions are randomly chosen as P-positions while the rest remain N-positions. In this case, we do not initially have a contiguous chunk of filled diagonals. However, as we proceed, we eventually reach a stage where a contiguous chunk of filled diagonals emerges.

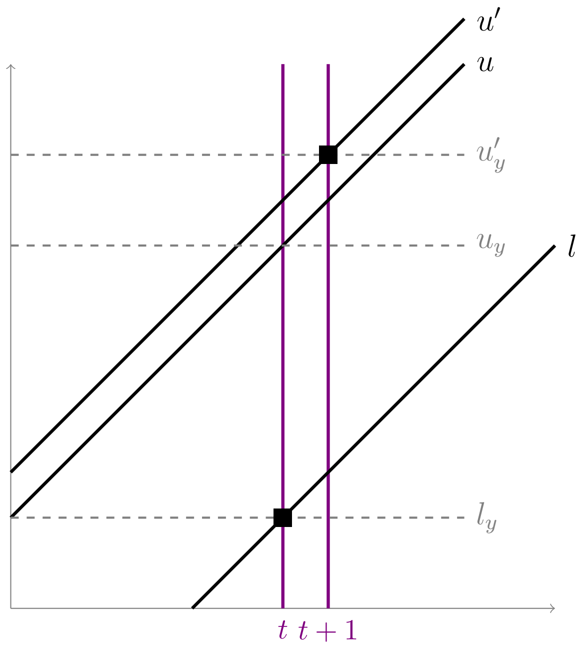

Figure 4.3: Case 1: when the P-position is on l at time t

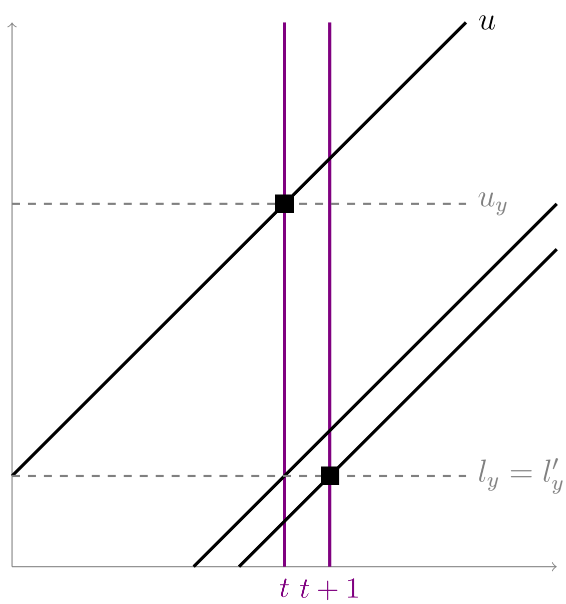

Figure 4.4: Case 2: i) when the P-position is on u at time t and l_y was not previously filled

From this point onward, we can systematically expand one of the diagonals each time a new P-position is identified. To accommodate this scenario, we will adapt Theorem 4.1.1 accordingly.

Theorem 10. *Suppose a fraction of all positions in the initial $k \times k$ box are randomly selected as P-positions. Let t represent the current column or time, $u = x + c_u$ denote the topmost filled diagonal, and $l = x + c_l$ denote the lowest filled diagonal. Additionally, let l_y represent*

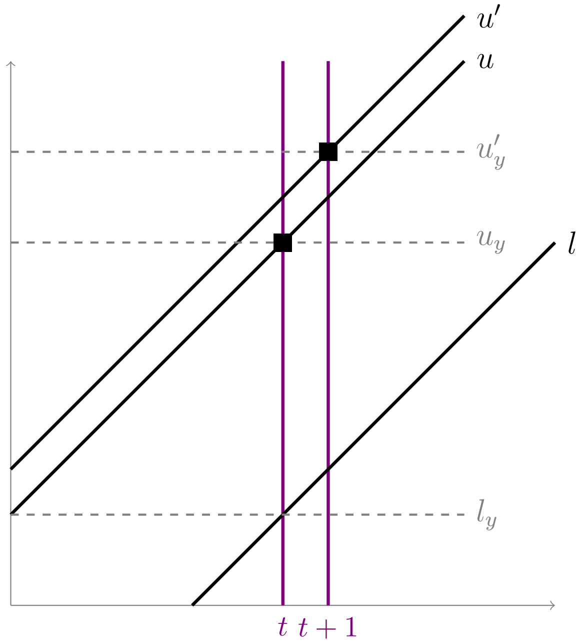

Figure 4.5: Case 2: ii) when the P-position is on u at time t and l_y was previously filled

the y-coordinate of the intersection of l with column t, and u_y represent the y-coordinate of the intersection of u with column t. As we progress, the following conditions consistently hold:

1. *The diagonals between l and u (inclusive) may not all be filled initially, but eventually, all diagonals in this range will be filled.*
2. *All rows below $l_y - 1$ are filled.*
3. *None of the rows above u_y are filled.*

This theorem parallels the previous one, albeit with a slight difference: there is no guarantee that all diagonals between l and u will be filled at any given time. However, eventually, this saturated phase will be reached, allowing us to proceed by expanding either l or u each time a P-position is encountered, as previously described. Once the saturated phase is reached, we can just apply the same proof as the previous theorem. At any given time, if the P-position on the previous column is on l, then we show there will be an unfilled diagonal between l and u that will be filled in the future. Furthermore, we show that this phenomenon (the P-position appearing on l) has to take place regularly, which implies eventually all the diagonals between l and u will be filled.

Proof. We aim to prove that eventually, we will reach the saturated state, at which point all conditions of Theorem 3.1.1 will be met, and we can proceed using the same proof from there.

The base case closely resembles the previous proof, except for the absence of the condition requiring all diagonals between l and u to be filled. In the induction step, at each time step, we either expand l, fill a diagonal between l and u, or expand u.

Case 1: If the P-position at time t is on l, then l cannot expand to l' at $t+1$ since we cannot have two P-positions on l_y. Consequently, the P-position at $t+1$ is on u'. At this point, there

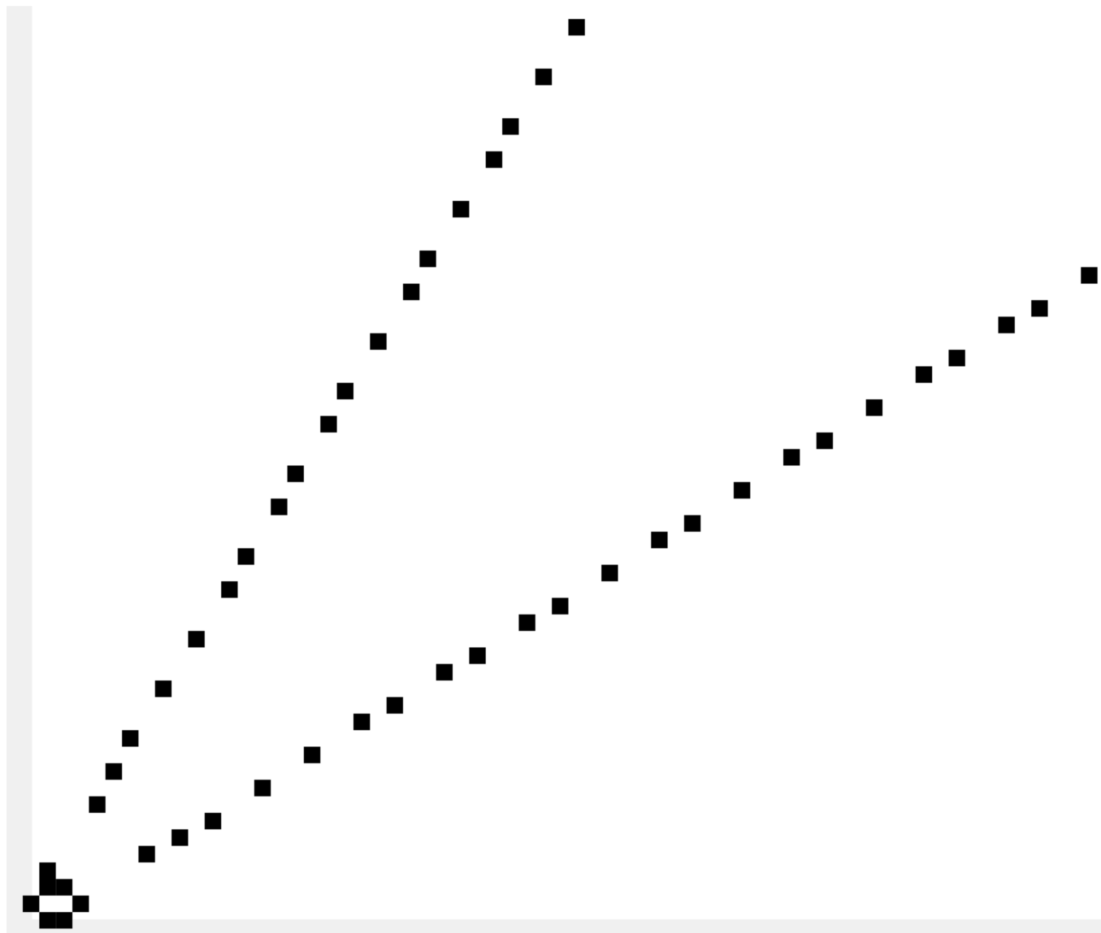

Figure 4.6: Randomly select 60% of the initial 6×6 positions as P-positions

must be at least two empty rows between the two P-positions directly below u_y, as illustrated in the first case of the previous theorem. Eventually, we need to fill these rows. We do so by filling the nearest unfilled diagonals between l and u since any diagonals above u do not intersect these rows in the future, and any diagonals below l intersect these rows later than the middle diagonals. Thus, every occurrence of this case results in filling at least one diagonal between l and u in a future time step.

Case 2: If the P-position at time t is not on l, then one of the following two cases must occur:

- We filled a diagonal between l and u at time t.
- We expand u, resulting in all positions between l_y and u_y being on either a filled row or a filled diagonal. Since there are empty diagonals between l and u, all rows between l and u are filled. However, each time we expand u to u', we create at least one empty row directly below u'_y. Similar to Case 1, we eventually fill these empty rows by filling a diagonal between l and u.

In all cases, at each time step, we either fill a diagonal between l and u, or we expand either l or u and must fill a diagonal between them later. Consequently, we complete filling all diagonals between l and u within finitely many time steps, reaching the saturated phase. □

4.2 Bit Representations of a Column

Consider representing each position/row between l_y and u_y (inclusive) at time t as a bit, where 1 indicates that the row is filled and 0 otherwise. So we have a column of bits. Then, at any given time t after all the diagonals between l and u are filled (let's call this stage "saturated phase"), the P-position at time t is either on l or u, which means at least one of l_y and u_y has bit 1. Let's see how the bit patterns change going from column t to $t+1$ by doing the same analysis as we did in expanding the diagonals.

- If the bottom bit is 1, then l_y has been filled at t or before t. Either way, u expands to u', so we add the bits 0 and 1 to the top (1 is the bit representing u'_y and 0 is the bit representing the row below it as $u'_y = u_y + 2$). We also erase the bottom bit since $l'_y = l_y + 1$; note that $l' = l$.
- If the bottom bit is 0, meaning that l_y was not filled, then the P-position at t is on u. So we fill l_y at $t+1$, causing l to expand to l'. Since $l'_y = l_y$, we change the bottom bit to 1. Furthermore, we add 0 to the top since $u'_y = u_y + 1$ is not filled.

In summary, if the bottom bit is 1, we erase it and add the bits 0 and 1 in order to the top. If the bottom bit is 0, we change it to 1 and add bit 0 to the top. At each step, the bit string becomes one bit longer.

4.2.1 Slope Calculation

We have the following bit rules:

Bottom bit	Action
0	Change the bottom bit to 1 and add 0 to the top
1	Erase the bottom bit and add bits 01 to the top

Note that if the bottom bit is 0, we change it to 1 first then apply the rule for when the bottom bit is 1. Then we can condense the two steps as one. The new rules become: Add bits 0, 0, 1 to the top if the bottom bit is 0; and add bits 0, 1 to the top if the bottom bit is 1. Then erase the bottom bit regardless of what it is. We update our bit rules to be the following:

Bottom bit	Action
0	Erase the bottom bit and add bits 001 to the top
1	Erase the bottom bit and add bits 01 to the top

When we change the bottom bit from 0 to 1 at time t, that means a new P-position is identified on that row at time t. Then we can obtain the slope of the lower line by looking at how often this phenomenon takes place.

We can do the similar thing to calculate the slope of the upper line. An interesting thing to note is that the topmost bit is always 1 since we either add 01 or 001 each time. Then going from one P-position to the next on the upper line just involves going from the previous top 1 bit to the next top 1 bit. Either we add 001 in two steps, or we add 01 in one step. In the first case, we go from one P-position to the next in 3 steps in the vertical direction, and in 2 steps in the horizontal direction. In the second case, we go from one P-position to the next in 2 steps in the vertical direction, and in 1 step in the horizontal direction. Then we can get the slope of the upper line if we know the fraction of 001 and 01 and take the weighted average, which is also the fraction of 0 and 1 respectively.

We define a as the bit pattern "001" and b as the bit pattern "01". When a is applied, it generates "aab", and when b is applied, it generates "ab". This mirrors how bit 0 generates "001" and bit 1 generates "01". Over time, the string evolves to consist of patterns "001" and "01", with the asymptotic density of "001" matching that of 0, and the asymptotic density of "01" matching that of 1.

To investigate the long-term fraction of 0 and 1 in a column, let p represent the asymptotic density of "001" and $1 - p$ represent the asymptotic density of "01". Let p' represent the asymptotic density of 0 bits. Each "001" pattern yields two 0 bits and one 1 bit, while each "01" pattern yields one 0 bit and one 1 bit. Thus: Then:

$$p' = \frac{2p + (1-p)}{3p + 2(1-p)} = \frac{1+p}{2+p}.$$

Since the asymptotic density of 0 bits converges to that of "001", we equate p' with p. Solving the equation yields $p = \frac{1}{\phi}$.

Now, let's revisit calculating the slope of the upper line. When we add "001" to the top, a new P-position is identified after two steps, and we ascend by 3 steps; when we add "01" to the top, a new P-position is identified after one step, and we ascend by two steps. Hence, the slope represents the average ratio of vertical steps to horizontal steps:

$$m = \frac{3p + 2(1-p)}{2p + (1-p)} = \frac{2+p}{1+p} = 1 + \frac{1}{\phi} = \phi.$$

This confirms that the slope remains consistent with the regular Wythoff game, as we hypothesized.

4.3 Future Work

For future research, we aim to explore the asymptotic behaviors of concurrently running two arbitrary binary bit strings of equal length under the defined bit rules. We assert that, after a finite number of time steps, both strings will maintain equal lengths and exhibit identical counts of 0s and 1s. Building on this observation, we introduce the following conjecture:

Conjecture 2. *After a finite number of time steps, once all diagonals between l and u have been filled, the column of bits will only exhibit the patterns 001 and 01. Comparing this column with the shifted column in Wythoff, aligned by the rows, there may be d discrepancies. The number of discrepancies will persist indefinitely; it neither increases nor decreases over time.*

This conjecture suggests that as time progresses, the graphs of our variant will closely resemble the graph of regular Wythoff Nim. To illustrate this, let's consider shifting the columns in the regular Wythoff Nim by an offset so that the lengths of its bit strings align with those of the variant after the saturated phase. At the beginning of the post-saturated phase, we have two binary strings of the same lengths: one for the variant game and the other for the regular Wythoff game that has been shifted. If the number of discrepancies remains constant, and the length of the two bit strings increases over time, the similarity between the two bit strings becomes more pronounced as time progresses.

Chapter 5

Conclusion

In conclusion, this thesis has explored Wythoff Nim and its variants, providing insights into their combinatorial properties and strategic implications. Standard Wythoff Nim displays symmetric lines of P-positions with slopes linked to the golden ratio. Conversely, the modified game $W(a, b)$, which constrains moves to multiples of specific numbers from each pile, introduces asymmetry and complexity while retaining two P-position lines. In the variant where positions are randomly designated as P- or N-positions, the P-position lines converge to Wythoff Nim's slopes, though without exact overlap. By associating the patterns of this variant with bit strings and adhering to a specific set of elongation rules, the similarity between the two games grows over time. Through a comprehensive analysis of P-position structures in both variants, we deepen our understanding of their gameplay dynamics.

Bibliography

[1] Samuel Beatty. Problem 3173. *The American Mathematical Monthly*, 34(3):159, 1927. 4

[2] Charles Leonard Bouton. Nim, a game with a complete mathematical theory. *Annals of Mathematics*, 3(1-4):35–39, 1902. doi: 10.2307/1967631. 1

[3] Ron Ferguson. Combinatorial game theory. *Carnegie Mellon University*, 2005. URL `https://www.cs.cmu.edu/afs/cs/academic/class/15859-s05/www/ferguson/comb.pdf`. 1

[4] Eric Friedman, Scott M Garrabrant, ILONA K PHIPPS-MORGAN, AS LANDSBERG, and Urban Larsson. Geometric analysis of a generalized wythoff game. *Games of No Chance 5*, 5:343, 2019. 3.2.1

[5] Urban Larsson. A generalized diagonal wythoff nim. *ArXiv preprint arXiv:1005.1555*, 2010. 3.2

[6] Carnegie Mellon University. Games 1 tutorial - combinatorial game theory, 2014. URL `https://contest.cs.cmu.edu/295/tutorials/games1-spring-2014-6up.pdf`. 1, 1, 1, 2.1

[7] Willem A Wythoff. A modification of the game of nim. *Nieuw Arch. Wisk*, 7(2):199–202, 1907. 1, 2.2, 2, 2.2, 3, 2.2