Reflected Number Systems

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Abstract

This paper introduces the class of *reflected number systems*. Members of this class have the adjacency property, i.e. representations for consecutive numbers differ in only one digit and the magnitude of this difference is one. It shows that every number system based on a positional weighting scheme has a reflected analogue. It presents procedures to convert a number system to its reflected analogue and vice versa. A conventional number system and its reflected analogue share some properties. Hence the conversion procedure can be used to design reflected number systems that are likely to have desired properties. This paper presents two examples.

Keywords: reflected number systems, Gray code, hamming distance, cyclic codes, positional weighting schemes, design of number systems

1 Introduction

The hamming distance between representations of consecutive numbers in conventional number systems can be anywhere between 1 and n, where n is the width of the representation. Many algorithms and applications, such as encoding input/output quantities, genetic algorithms, graph embedding, hypercube algorithms including data distribution, emulation of other networks and processor allocation, signal transmission and library document ordering, require that the hamming distance between representations of consecutive numbers be exactly one. Gray code[3] is the best-known example of representations that satisfy this requirement (Such representations are called *cyclic* [3]) and has been widely used. Such representations are usually generated in an ad-hoc manner, as an arbitrary set of digit sequences. No cyclic number systems had been developed till recently when Acharya [1] and Irshid [2] independently provided a weighting scheme for the Gray code.

The Gray code is a member of the class of *reflected number systems*. Members of this class possess the adjacency property, i.e representations for consecutive numbers differ in only one digit and the magnitude of this difference is one. This is a stronger version of the cyclic constraint which says the hamming distance between consecutive numbers should be exactly one. These number systems are called *reflected* because it is possible to generate representations for the set of n-digit numbers by *reflecting* the representations for the set of $(n-1)$-digit numbers. Figure 1 shows three examples. The horizontal bars in the figure represent reflections.

Section 2 describes reflected representations in greater detail and show that they cyclic constraint. It also shows that every conventional number system which represents numbers as a weighted sequence of digits has a reflected analogue. Section 3 describes weighting schemes for reflected representations. Section 4 describes the design of reflected number systems that are likely to have specific properties. It presents two examples for the properties "division by powers of two is equivalent to right shifts" and "easy formation of additive inverse".

2 Reflected Representations

A conventional number system that is based on a positional weighting scheme and a digit set D generates representations for numbers in its range by iterating over D for each position. Iteration for each position is performed in a standard order imposed on D. Iterations for different positions proceed at different rates, the rate for each position proceeds slower than that for the position to its right. The ratio of the rates at successive positions is equal to the $|D|$, i.e. the digit at position i is changed once after a complete iteration at position $(i-1)$. When one iteration is completed for a particular position, the next one begins with the same order on D. This procedure leads to *hamming cliffs*[1] between the number at the end of one iteration and the number at the beginning of the next one. Several examples of this are shown in Figure 2. In the figure, hamming cliffs are indicated by underlining.

These hamming cliffs can be avoided if the order on D is reversed at the end of each iteration. Consider positions i and $(i-1)$. When an iteration ends at position $(i-1)$, the digit at that position is the last one in the current order. When the next iteration begins at position $(i-1)$,

[1] Hamming cliffs occur between sequences of digits that differ in more than one position.

Number	Gray	Ternary	Signed-digit ternary
0	0000	00	$\overline{1}\overline{1}$
1	0001	01	$\overline{1}0$
2	0011	02	$\overline{1}1$
3	0010	12	01
4	0110	11	00
5	0111	10	$0\overline{1}$
6	0101	20	$1\overline{1}$
7	0100	21	10
8	1100	22	11
9	1101		
10	1111		
11	1110		
12	1010		
13	1011		
14	1001		
15	1000		

Figure 1: Examples of reflected number systems

the digit at position i is replaced by the next digit in the current order for that position. As the order for position $(i-1)$ has been reversed, the digit at that position does not change. Hence, the digit sequences differ in only one position. Examples of this can be seen in Figure 1. Since the order of digits is reversed, or *reflected*, at the end of each iteration, such representations are called *reflected representations*. Reflected representations satisfy a stronger version of the cyclic constraint. Not only do the representations of consecutive numbers differ at only one position, but the magnitude of the difference between the digits at that position is exactly one.[2]

Note that the procedure described in the previous paragraph provides algorithms to convert the representations of numbers in a conventional number system to their reflected counterparts and vice versa . Note also that it does not depend on the any particular parameter of the number system. The only assumption that has been made is that the conventional number system is based on a positional weighting scheme. There is no constraint on the weights themselves, the digits used in the representations or the width of the representations. As a result, the algorithms are applicable to all conventional number systems based on positional weighting schemes. This includes positive, negative and imaginary radix number systems, radix-complement number systems, number systems based on an arbitrary sequence of weights, number systems with standard and non-standard digit sets etc. An important class

[2]To be more rigorous, the digits at that position are next to each other in the standard order on D.

Number	Binary Representation
0	000
1	0<u>01</u>
2	0<u>10</u>
3	<u>011</u>
4	<u>100</u>
5	1<u>01</u>
6	1<u>10</u>
7	111

Figure 2: Hamming cliffs in 3-bit binary number system

of number systems to which it is not applicable is the class of *residue number systems* the members of which are not based on positional weighting schemes.

3 Reflected Number Systems

In a reflected representation, the order of the digits is reversed in each iteration. Therefore, the value of a sequence of digits depends on the number of reflections at each position (this is the same as the number of completed iterations of the D at that position). Since there are only two possible orders, the value actually depends only on the oddness of number of iterations at each position and not its magnitude.

Consider the number N within the range of a reflected representation. Since the value of N can be represented as a summation of the value of the digits at individual positions, consider the value of the i^{th} digit, x_i. As discussed in the previous section, order for the i^{th} digit is changed after every $|D|^i$ numbers. Therefore, The number of iterations at the i^{th} position is equal to the value of $(x_{n-1}, x_{n-2}, ..., x_{i+1})$. If this value is odd, then x_i is from the standard order, else it is from the reversed order.

A conventional number system can be characterized by the tuple (n, D, W_n), where n is the width of the representations, D is the set of digits used in the representations and W_n is the sequence of weights, $w_i, i \in \{0, ..., n-1\}$. The value of a sequence of digits, $(x_{n-1}, x_{n-2}, ..., x_1, x_0)$ in the number system is given by the formula $X = \sum_{j=0}^{n-1} x_j w_j$

Let $d_l, l \in \{0, ..., k-1\}$ be the standard order on D. Then the *conjugate* order is defined as $d^c{}_l, l \in \{0, ..., k-1\} : d^c{}_l = d_{(k-1-l)}$. The value of a sequence of digits in the reflected analogue of (n, D, W_n) is given by the formula:

$$X = \sum_{j=0}^{n-1} v_j w_j \tag{1}$$

$$v_j = \lambda_j x_j^c + \lambda_j^c x_j \tag{2}$$

$$x_j^c = conjugate(x_j) \tag{3}$$

$$\lambda_j = odd(x_{n-1}, x_{n-2}, ..., x_{j+1}) \tag{4}$$

$$\lambda_{n-1} = 0 \tag{5}$$

$$\lambda_j^c = 1 - \lambda_j \tag{6}$$

The conjugate of a digit d_i in the standard order is the digit d_i^c in the conjugate order. λ_j is the reflection parity of the j^{th} position, i.e. the oddness of the number of iterations at the j^{th} position. It can take the values 0 and 1. It can be recursively defined as:

$$\begin{aligned} \lambda_j &= \lambda_{j+1} \oplus odd(x_{j+1}) \\ \lambda_{n-1} &= 0 \\ odd(x_{j+1}) = odd(l) &: x_{j+1} = d_l \end{aligned}$$

The equation above means that the oddness of a digit is defined as the oddness of the position it occupies in the standard order. This allows the set of equations to hold for arbitrary digit sets.

4 Designing Reflected Number Systems

A conventional number system and its reflected analogue share all the three parameters of the 3-tuple (n, D, W_n) that characterizes a number system based on positional weights. The difference is primarily in the sequence of digits that are used to represent particular numbers. As a result, they share those properties that do not depend on the sequence of digits used to represent particular numbers. Examples include aggregate properties like range, redundancy, completeness, largest number representable, smallest number representable etc. On the other hand, properties that depend on the sequence of digits used to represent particular number may or may not be shared depending on the details of the particular number system and the property. Examples of such properties include whether the oddness of the number can be determined by examining the last digit, whether division by powers of two is equivalent to right shifting, fast generation of additive inverse, multiple representations for zero, number of occurrences of a particular digit in the digit sequence etc.

To design a reflected number system with a paricular property that does not depend on the digit sequences used for particular numbers, find a conventional number system which has this property and generate its reflected analogue. Design of reflected number systems with properties that depend on the digit sequences used for particular numbers has a generate-and-test flavor since it may be necessary to find several conventional number systems with the particular property before finding one for which the property is preserved across the conversion. In fact, no such number system may exist. An example of such a property is "equivalence of simple left shifts with multiplication with powers of two". No reflected number system can have this property because shifting the most significant bit out might change the reflection parity for all positions and the value of the least significant bit shifted in depends on the value of the number being shifted.

Following subsections briefly illustrate the design of two reflected number systems.

Number	Original	Shifted
0	000	000
1	001	000
2	011	001
3	010	001
4	110	011
5	111	011
6	101	010
7	100	010

Figure 3: Right shifts on the 3-bit gray number system

4.1 The Gray Number System

Suppose the property of interest is *division by powers of two is equivalent to right shifting*. A conventional number system for which this property holds is the binary number system. The reflected analogue of the binary number system is the gray number system. It can easily be shown that the gray number system has the desired property. Figure 3 provides an illustration.

4.2 The Reflected Two's Complement Number System

Suppose the property of interest is *easy generation of additive inverse*. A conventional number system that has this property is the two's complement number system. The reflected analogue of the two's complement number system is the reflected two's complement number system. The following derivation shows that the desired property holds for it. To generate the additive inverse of a reflected two's complement number, complement the most significant bit and add one to the resulting number. In the following derivation, X^c is the number generated by performing this procedure.

$$
\begin{aligned}
X = &\quad -x_{n-1}2^{n-1} + \textstyle\sum_{j=0}^{n-2}(\lambda_j x_j^c + \lambda_j^c x_j)2^j \\
-X = &\quad x_{n-1}2^{n-1} - \textstyle\sum_{j=0}^{n-2}(\lambda_j x_j^c + \lambda_j^c x_j)2^j \\
= &\quad -(1 - x_{n-1})2^{n-1} + 2^{n-1} - \textstyle\sum_{j=0}^{n-2}(\lambda_j x_j^c + \lambda_j^c x_j)2^j \\
= &\quad -x_{n-1}^c 2^{n-1} + \textstyle\sum_{j=0}^{n-2} 2^j + 1 - \sum_{j=0}^{n-2}(\lambda_j x_j^c + \lambda_j^c x_j)2^j \\
= &\quad -x_{n-1}^c 2^{n-1} + \textstyle\sum_{j=0}^{n-2}(1 - (\lambda_j x_j^c + \lambda_j^c x_j))2^j + 1 \\
= &\quad -x_{n-1}^c 2^{n-1} + \textstyle\sum_{j=0}^{n-2}(\lambda_j^c x_j^c + \lambda_j x_j))2^j + 1 \\
X^c = &\quad -x_{n-1}^c 2^{n-1} + \textstyle\sum_{j=0}^{n-2}(\lambda_j^c x_j^c + \lambda_j x_j))2^j + 1
\end{aligned}
$$

Note that complementing the most significant bit complements all the λ_j. Figure 4 shows the 3-bit reflected two's complement number system.

Number	Two's complement	Reflected two's complement
-4	100	110
-3	101	111
-2	110	101
-1	111	100
0	000	000
1	001	001
2	010	011
3	011	010

Figure 4: The 3-bit reflected two's complement number system

References

[1] A. Acharya. Reflected number systems, Sept 1986. All India Student Paper Contest, IIT Kharagpur.

[2] M. I. Irshid. Gray code weighting system. *IEEE Transactions on Information Theory*, 33(6):930–931, Nov 1987.

[3] Z. Kohavi. *Switching and Automata Theory*. Tata McGraw-Hill, New Delhi, 1978.