LEVEL SPACINGS FOR $SL(2, p)$

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Abstract. We investigate the eigenvalue spacing distributions for randomly generated 4-regular Cayley graphs on $SL_2(\mathbb{F}_p)$ by numerically calculating their spectra. We present strong evidence that the distributions are Poisson and hence do not follow the Gaussian orthogonal ensemble. Among the Cayley graphs of $SL_2(\mathbb{F}_p)$ we consider are the new expander graphs recently discovered by Y. Shalom. In addition, we use a Markov chain method to generate random 4-regular graphs, and observe that the average eigenvalue spacings are closely approximated by the Wigner surmise.

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1. Introduction. One of the most remarkable numerical discoveries of the recent past is Odlyzko's finding that the spacings of the zeros of the Riemann zeta function closely follow the Gaussian unitary ensemble of random matrix theory [15]. As a result of this work, attention has turned to the spacing distributions for the spectrum of other natural classes of operators, in the hope of making similar connections with other number theoretic objects.

One direction of related work is towards the analysis of the eigenvalue spacing distribution for the Laplacian on different manifolds; see [17] for a survey of many recent results of this type and an extensive bibliography. The motivation for our work comes from the particular case of interest of the Laplacian on $SL_2(\mathbb{Z})\backslash\mathbb{H}$, where $\mathbb{H}$ denotes the hyperbolic upper half plane. This is the main example of an *arithmetic surface*, which is a hyperbolic surface given as the quotient of the upper half plane by an arithmetic subgroup of $SL_2(\mathbb{R})$.

In general, the statistical behavior of eigenvalue spacings for natural families of operators falls into two main classes, Poisson and the Gaussian ensemble. The Gaussian orthogonal ensemble (GOE) governs random symmetric matrices while the Gaussian unitary ensemble (GUE) applies to random complex Hermitian matrices. The density of the spacings in the Poisson case, where the spacings are normalized to have unit mean, is exponential, e^{-x}, and for the Gaussian orthogonal ensemble the density is well-approximated by the *Wigner surmise* $\frac{\pi s}{2}e^{-\pi x^2/4}$ [13]. More generally, Katz and Sarnak [7] have recently investigated the eigenvalue spacings for the classical groups, as well as connections to zeta functions for curves over finite fields. Diaconis and Shahshahani [3] have analyzed the eigenvalue distribution for a randomly chosen matrix from classical groups. The physical interpretation of eigenvalues is as energy levels, and Poisson behavior of the spacings is usually thought of as characteristic of integrable systems, while GOE corresponds to chaotic systems (cf. [1,4,13] and the many references therein).

Computations of Schmit [19] indicate that the spacing distribution for $SL_2(\mathbb{Z})\backslash\mathbb{H}$ should be Poisson. More generally, recent work of Luo and Sarnak [12] and Rudnick and Sarnak [16] indicates that this is the case for any arithmetic surface. The exact behavior of the spacings in this case is still an open question.

In this paper we investigate the eigenvalue spacings for the adjacency matrices of various Cayley graphs on $SL_2(\mathbb{F}_p)$, naturally thought of as discrete approximations to the spectral behavior in the continuous setting of $SL_2(\mathbb{Z})\backslash\mathbb{H}$. This sort of analogy is suggested by the machinery developed in the successful application of Selberg's Theorem [20] to the discovery of expander graphs built as Cayley graphs of $SL_2(\mathbb{F}_p)$. Here the main tool is the transfer of the lower bound on the first positive eigenvalue of the Laplacian to a uniform bound on the first nonzero eigenvalue for a family of graphs obtained as quotients of Cayley graphs on $SL_2(\mathbb{Z})$. See [10] for an excellent treatment of this construction and a thorough bibliography.

The results in this paper are of two types. On the one hand we compute the spacing distributions for particular Cayley graphs on $SL_2(\mathbb{F}_p)$. These computations include generator sets to which Selberg's Theorem would apply, as well as some sets to which it would not apply. We also compute the distributions for random 4-regular Cayley graphs on $SL_2(\mathbb{F}_p)$ (for a fixed prime p). In all cases we find close agreement with Poisson behavior. For comparison we have also computed the spectra for randomly generated 4-regular graphs, using Markov chain methods to generate the graphs. These all closely follow the GOE, and this is in agreement with the extensive computations performed by Jakobson, Miller, Rivin and Rudnick [5]. Our computations on $SL_2(\mathbb{F}_p)$ are only made possible by the use of representation theory for the groups $SL_2(\mathbb{F}_p)$ (see Section 2). In brief, we compute the spectrum as the union of spectra of individual Fourier transforms of the characteristic function of the generating set. The second type of computation we carry out investigates the spectra of some of these individual transforms. This includes the spectrum of the expander graphs built as the action of $SL_2(\mathbb{F}_p)$ on the projective line. Again, in all cases the distributions appear to exhibit Poisson behavior.

After outlining our use of Fourier analysis in Section 2, we present our computations in Section 3. We summarize our results as two conjectures in Section 4.

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2. Cayley graphs and Fourier analysis. As in [8,9], to analyze the spectrum of Cayley graphs we exploit the fact that the adjacency matrix can be viewed as the Fourier transform of the generators at the right regular representation. Any representation is equivalent to a direct sum of irreducible representations. Since we are able to compute Fourier transforms at any irreducible representation for $SL_2(\mathbb{F}_p)$, we are able to recover the complete spectrum by only computing the spectrum of the individual Fourier transforms. For $SL_2(\mathbb{F}_p)$, all multidimensional representations are roughly of degree p, versus $O(p^3)$, the size of the regular representation.

To say this more precisely, let G be a finite group and let $\mathcal{S} \subset G$ generate G. The *Cayley graph* $X = X(G, \mathcal{S})$ for G with respect to $\mathcal{S}$ is the undirected graph with vertex set equal to G, such that there is an edge between a and b in X if and only if $as = b$ for some $s \in \mathcal{S} \cup \mathcal{S}^{-1}$. Equivalently, the adjacency matrix of $X(G, \mathcal{S})$ has a one in the (a, b) entry if and only if $as = b$ for some $s \in \mathcal{S} \cup \mathcal{S}^{-1}$. Let ρ_{reg} denote the right regular representation of G, computed with respect to the basis of delta functions on G. Then it is not difficult to see we have the following expression for the adjacency matrix of $X(G, \mathcal{S})$, denoted $\Gamma(G, \mathcal{S})$:

$$\Gamma(G, \mathcal{S}) = \sum_{s \in \mathcal{S} \cup \mathcal{S}^{-1}} \rho_{reg}(s). \tag{2.1}$$

The righthand side of (2.1) is also the *Fourier transform* of the characteristic function for $\mathcal{S} \cup \mathcal{S}^{-1}$.

Direct computation of the spectrum of the $|G| \times |G|$ matrix $\Gamma(G, \mathcal{S})$ requires $O(|G|^3)$ operations (cf. [24]). For $SL_2(\mathbb{F}_p)$, since $|SL_2(\mathbb{F}_p)| = O(p^3)$, this means $O(p^9)$ operations. This cost quickly becomes prohibitive. However, by using the tools of representation theory, we may instead compute the elements of an equivalent block diagonal matrix and realize the entire spectrum as the union of the spectra of the blocks.

More precisely, representation theory gives a simultaneous block diagonalization of the matrices $\rho_{reg}(s)$ as

$$\rho_{reg}(s) \sim \begin{pmatrix} B_1(s) & 0 & \cdots & 0 \\ 0 & B_2(s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_h(s) \end{pmatrix} \tag{2.2}$$

with

$$B_i(s) = \begin{pmatrix} \rho_i(s) & 0 & \cdots & 0 \\ 0 & \rho_i(s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \rho_i(s) \end{pmatrix} \tag{2.3}$$

where $\rho_1, \ldots, \rho_h$ are a complete set of irreducible matrix representations of G with $\deg(\rho_i)$ copies of $\rho_i(s)$ on the diagonal. Consequently,

$$\mathit{spectrum}(X(G, \mathcal{S})) = \bigcup_{i=1}^{h} \mathit{spectrum}\left(\sum_{s \in \mathcal{S} \cup \mathcal{S}^{-1}} \rho_i(s) \right). \tag{2.4}$$

The degree of the largest irreducible representation of G is bounded above by $|G|^{1/2}$ (cf. [21]). Thus, by using the basis which gives (2.2) and (2.3), we are able to reduce the computation from $O(|G|^3)$ operations to a potentially more manageable $O\big(|G| \cdot (\max_i \deg(\rho_i))\big) \leq O(|G|^{3/2})$ operations.

For $G = SL_2(\mathbb{F}_p)$, the irreducible representations of $SL_2(\mathbb{F}_p)$ occur in two families, the *discrete series* and *principal series*. The distinction depends upon the restriction of an irreducible representation to the Borel subgroup $B < SL_2(\mathbb{F}_p)$ of upper triangular matrices. An irreducible representation of $SL_2(\mathbb{F}_p)$ is said to be from the principal series if its restriction to B contains the trivial representation. Otherwise, it is said to be from the discrete series. The principal series representations occur as components of induced 1-dimensional representations from B, which are almost always irreducible. This gives the trivial representation, one representation of degree p, two representations of degree $(p+1)/2$, and $(p-3)/2$ representations of degree $p+1$. The discrete series is less easily explained, but suffice it to say here that the representations are in close correspondence to the characters of the non-split torus in $SL_2(\mathbb{F}_p)$ (cf. [14], Ch. 2, Section 5). There are two such representations of degree $(p-1)/2$ and $(p-1)/2$ representations of degree $p-1$.

Explicit representations are needed to apply (2.2) and (2.3). These can be found in [14] and [23] and are the basis of our implementation (cf. [8]). Knowledge of the representations of $SL_2(\mathbb{F}_p)$ gives the irreducible representations of $PSL_2(\mathbb{F}_p)$. More precisely, if ρ is an irreducible matrix representation of $SL_2(\mathbb{F}_p)$ and $-I$ is in the kernel of ρ, where $I = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$, then ρ is constant on cosets $SL_2(\mathbb{F}_p)/\{\pm I\}$ and as such gives an irreducible representation of $PSL_2(\mathbb{F}_p)$. Under this identification, the set $\{\rho \mid \{\pm I\} \subset \ker(\rho)\}$ gives a complete set of inequivalent irreducible representations of $PSL_2(\mathbb{F}_p)$. We use this correspondence when calculating the spectra for the Ramanujan graphs constructed by Lubotzky, Phillips and Sarnak in Section 3.

Remark. In the case in which $\mathcal{S} \cup \mathcal{S}^{-1}$ is a union of conjugacy of classes of G, the adjacency matrix can be completely diagonalized and the eigenvalues can be computed as certain character sums over G [2].

3. Numerical evidence. The main results of this note are computations of the spacings for various Cayley graphs on $SL_2(\mathbb{F}_p)$, made possible by the techniques outlined in Section 2. As we stated in the introduction, our computations are of two types:

(1) The computation of spacing distributions for particular 2-generator Cayley graphs on $SL_2(\mathbb{F}_p)$.
(2) The computation of spacing distributions for particular Fourier transforms for 2-generator Cayley graphs on $SL_2(\mathbb{F}_p)$.

Within these computations we distinguish among various kinds of generators; *global* or *Selberg* generators are generators for families of Cayley graphs obtained as projections of a single Cayley graph on $SL_2(\mathbb{Z})$; *non-Selberg* generators are an infinite family of generators for $SL_2(\mathbb{F}_p)$ as $p \to \infty$ that are not the projection of a single set of generators for $SL_2(\mathbb{Z})$; *random* generators are a pair of generators for $SL_2(\mathbb{F}_p)$ generated by a simple randomized algorithm. In general, random generators are non-Selberg.

For a generating subset $\mathcal{S} \subset SL_2(\mathbb{F}_p)$ we denote the eigenvalues (without multiplicities) of $\Gamma(G, \mathcal{S})$ as $\lambda_0 > \lambda_1 > \cdots > \lambda_N$, and let $P(S)$ denote the "empirical" cumulative distribution function for the eigenvalue spacings, so that

$$P(S) = \frac{1}{N}\sum_{j=1}^{N}[\lambda_{j-1} - \lambda_j \leq S] \tag{3.1}$$

where $[a \leq b]$ is one if $a \leq b$ is true and is zero otherwise. We assume the eigenvalues are normalized so that the spacings have mean one:

$$\frac{1}{N}\sum_{j=1}^{N}(\lambda_{j-1} - \lambda_j) = 1\,. \tag{3.2}$$

Besides the computations of the various $P(S)$, we also include some new data on a particularly interesting non-Selberg generating pair recently discovered by Y. Shalom [22]. These turn out to be (in terms of the numerical analysis of the second-largest eigenvalue) among the best expanders built as Cayley graphs of $SL_2(\mathbb{F}_p)$ discovered to date.

3.1. Random graphs. In this section we present sample results from our calculations of the spacing distributions of random 4-regular graphs, where the curves show GOE behavior (see also [5]), and 4-regular Cayley graphs of $SL_2(\mathbb{F}_p)$ for randomly chosen generating pairs, where the behavior is Poisson.

To choose generating pairs of $SL_2(\mathbb{F}_p)$ uniformly at random, we use the algorithm described in [8]. This algorithm first selects two group elements $a, b \in SL_2(\mathbb{F}_p)$ uniformly at random using the Bruhat decomposition of $SL_2(\mathbb{F}_p)$, and then checks whether $\{a, b\}$ generates the group by verifying that $\{\pi(a), \pi(b)\}$ does not generate one of the six possible subgroups of $PSL_2(\mathbb{F}_p)$, where $\pi : SL_2(\mathbb{F}_p) \to PSL_2(\mathbb{F}_p)$ is the natural projection. We refer to [8] for details.

We generated random Cayley graphs using this algorithm for $p \geq 150$, and observed that all of the graphs closely followed the exponential distribution. In Figure 3.1 we show the cumulative distribution function for a typical example.

Remark. In Figure 3.1, as in all of the cumulative distributions that we present, we have omitted the spectral gap $k - \lambda_1$ from the calculations. Asymptotically, as $p \to \infty$, this gap does not contribute to the cdf, although it changes the mean of the distribution.

To generate random 4-regular graphs (not Cayley graphs) we used a Markov chain method. In general, the states of the Markov chain are the (labeled) k-regular graphs, and two graphs can be connected by a single step of the random walk if and only if the symmetric difference of their edges is a cycle of length 4. The random walk can be described in terms of the incidence matrices of the graphs. Recall that the incidence matrix of a graph $\Gamma(E, V)$ is the $|V| \times |E|$ matrix $I(\Gamma)$ where the column corresponding

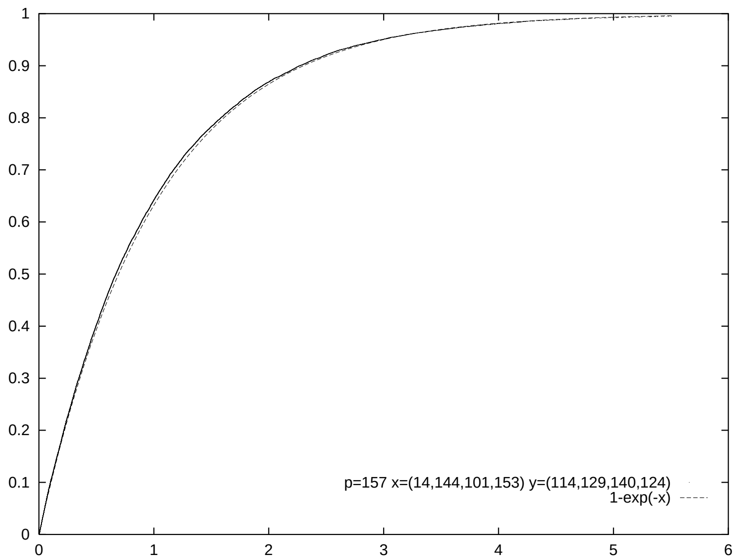

FIG. 3.1. *The cumulative distribution function $P(S)$ for the eigenvalue spacings of the single, randomly chosen generating pair $a = \begin{pmatrix} 14 & 144 \\ 101 & 153 \end{pmatrix}$, $b = \begin{pmatrix} 114 & 129 \\ 140 & 124 \end{pmatrix}$ for $SL_2(\mathbb{F}_{157})$. The curve does not include the spacing between the first and second eigenvalues, which in this case was $1 - 0.879090$. The dashed line is the curve $1 - e^{-x}$.*

to edge (i, j) has a 1 in the i^{th} and j^{th} row, and 0's elsewhere. If I is the state of the random walk, two rows $1 \leq i, j \leq |V|$ and two columns $1 \leq k, l \leq |E|$ are chosen uniformly at random. If $I_{ik} = I_{jl} = 1$ and $I_{il} = I_{jk} = 0$ then the chain moves to the state with $I_{ik} = I_{jl} = 0$ and $I_{il} = I_{jk} = 1$ unless a double edge is formed by doing so. Similarly, if $I_{ik} = I_{jl} = 0$ and $I_{il} = I_{jk} = 1$ then the chain moves to the state with $I_{ik} = I_{jl} = 1$ and $I_{il} = I_{jk} = 0$, again unless a double edge would be formed by this move. In all other cases the walk remains in the same state.

It is proved in [6] that this random walk is rapidly mixing, using the technique of canonical paths to estimate conductance. While this results in a polynomial algorithm in the size of the graphs, the exponent is too large to enable this result to yield a stopping criterion for the graphs we generate. As a matter of practicality, we simply run the chain for a large (10^8) number of steps to generate each graph. After stopping the chain, we test to make sure the graph is connected (it is with high probability).

Since the Cayley graphs for $SL_2(\mathbb{F}_{157})$ yield on the order of 25,000 eigenvalues and 19,000 intervals, we generated 10 random 4-regular graphs on 2,000 vertices and averaged the intervals to obtain comparable statistics. The resulting cumulative distribution function is shown in Figure 3.2, where it is compared with the Wigner surmise.

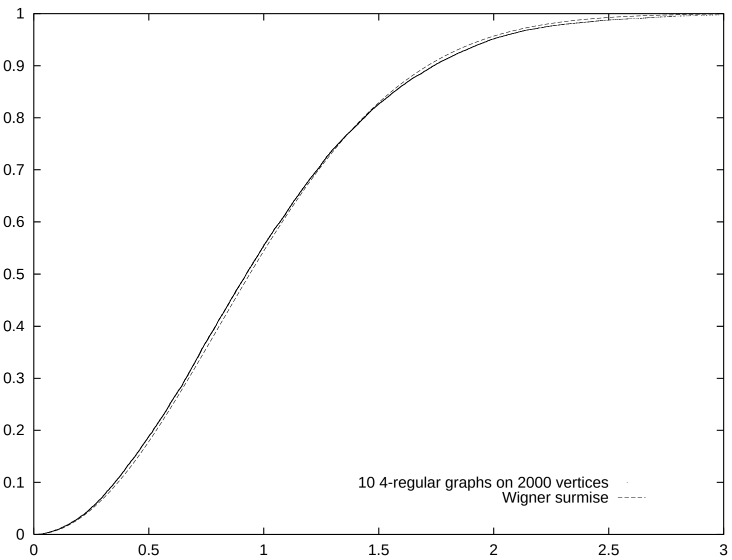

FIG. 3.2. *The cumulative distribution function $P(S)$ for the eigenvalue spacings of the 10 4-regular graphs on 2,000 vertices, generated by running the Markov chain for 10^8 steps. The average value of λ_1 was 0.863389. The dashed line is the cdf for the Wigner surmise,* $1 - \exp(-\frac{\pi x^2}{4})$.

3.2. Explicit generators. When we computed the spacing distributions for both Selberg and non-Selberg type generators, and for the generators recently discovered by Shalom [22], we found that they followed the Poisson behavior very closely. The generators for $SL_2(\mathbb{F}_p)$ we used were the following:

$$\left\{ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\} \qquad \textit{Selberg} \tag{3.3}$$

$$\left\{ \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \right\} \qquad \textit{non-Selberg} \tag{3.4}$$

$$\left\{ \begin{pmatrix} 1+\omega & -1 \\ -\omega & 1 \end{pmatrix}, \begin{pmatrix} -2\omega & -\omega \\ 1+\omega & 1+\omega \end{pmatrix} \right\} \qquad \textit{Shalom} \tag{3.5}$$

where ω is a primitive cube root of unity $(mod\ p)$. For $p = 199$, the distribution for Shalom's generators is shown in Figure 3.3. The curves for the other generators are very similar.

The second-largest eigenvalue λ_1 is shown for a few of the small primes $p = 1\ (mod\ 4)$ in Table 3.2. These numbers indicate that, except for the known Ramanujan graphs, these generators are perhaps the best explicit 4-regular graphs for $SL_2(\mathbb{F}_p)$ that have been obtained. For $PSL_2(\mathbb{F}_p)$, the explicit Ramanujan graphs of Lubotzky, Phillips and Sarnak have better separation. For comparison, the second-largest eigenvalue for the LPS gen-

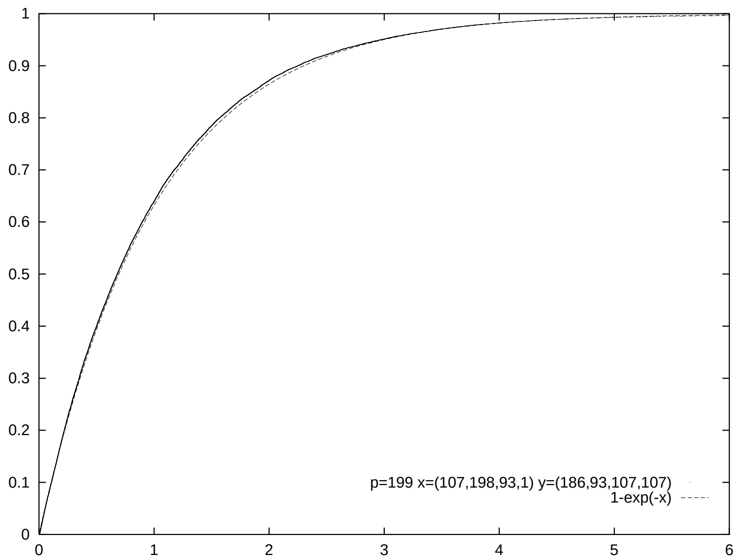

FIG. 3.3. *The cumulative distribution function $P(S)$ for generators (3.5) with $p = 199$, so $a = \left(\begin{smallmatrix} 107 & 198 \\ 93 & 1 \end{smallmatrix}\right)$, $b = \left(\begin{smallmatrix} 186 & 93 \\ 107 & 107 \end{smallmatrix}\right)$. The second-largest eigenvalue is $\lambda_1 \approx 0.886048$.*

erators

$$\frac{1}{\sqrt{3}}\begin{pmatrix} i & 1\pm i \\ -1\pm i & -i \end{pmatrix}, \quad \frac{1}{\sqrt{3}}\begin{pmatrix} i & -1\pm i \\ 1\pm i & -i \end{pmatrix} \tag{3.6}$$

where $i = \sqrt{-1}$, are shown for several values of $p = 3\ (mod\ 4)$ in Table 3.2.

Not all of the spacing distributions that we observed for $SL_2(\mathbb{F}_p)$ were so closely Poisson. As an example, the plot in Figure 3.4 shows the spacing distribution for the Selberg type generators

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\} \tag{3.7}$$

for $p = 199$. We believe that the discrepancy can be attributed to the expansion properties of the graph. (The difference cannot be attributed to gaps in the spectrum, since other than the gap $4 - \lambda_1$, the spectrum is "continuous.") The second-largest eigenvalue for this graph is $\lambda_1 \approx 0.977554$.

3.3. Individual transforms. The data presented above strongly indicates that the average spacing associated with the Fourier transforms of the delta function supported on the generating set is asymptotically Poisson. In this section we investigate the behavior of the individual transforms, and present sample calculations for the LPS graphs $X^{p,q}$ [11,18]. Of particular interest is the transform at the principal series representation induced

p	λ_1	p	λ_1
13	0.921721	109	0.908280
19	0.864292	127	0.886755
31	0.879056	139	0.887515
37	0.894603	151	0.880739
43	0.894659	157	0.888702
61	0.882493	163	0.881105
67	0.890386	181	0.890412
73	0.903236	193	0.887729
79	0.881429	199	0.886048
97	0.899836	211	0.881852
103	0.883868	223	0.882423

TABLE 3.1
λ_1 *for a sequence of Shalom's expanders.*

p	λ_1
13	0.832880
37	0.863086
61	0.865375
73	0.862093
97	0.864023
109	0.863180
157	0.861790
181	0.863598
193	0.862532
229	0.865491
241	0.864479

TABLE 3.2
λ_1 *for* 4*-regular Ramanujan graphs (3.6) on* $PSL_2(\mathbb{F}_p)$*. The Ramanujan bound is* $\sqrt{3}/2 \approx 0.8660254$.

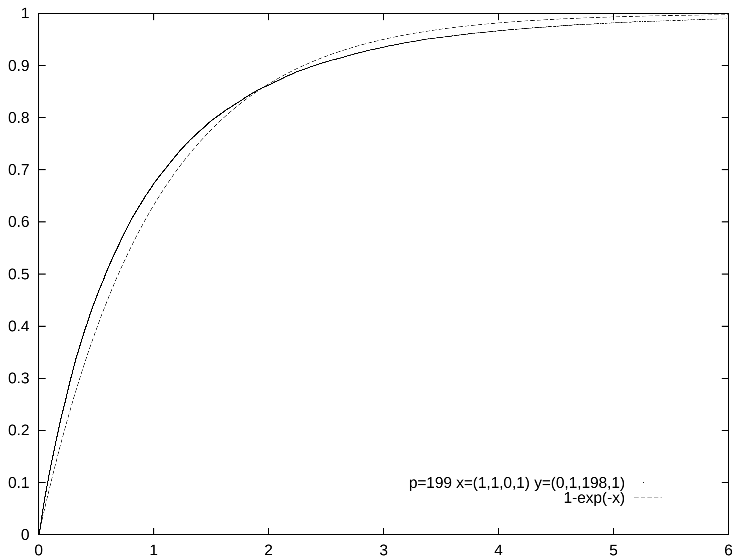

FIG. 3.4. *$P(S)$ with generators* $a=\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), b = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ *for* $SL_2(\mathbb{F}_{199})$. *The second-largest eigenvalue is* $\lambda_1 \approx 0.977554$.

from the identity, since this is associated with a graph on the projective line.

To explain, we recall that $SL_2(\mathbb{F}_q)$ acts on the projective line $\mathbb{P}^1(\mathbb{F}_q) = \{0, 1, \ldots, q-1, \infty\}$ by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \omega = \frac{a\,\omega + b}{c\,\omega + d}\,.$$

Let B denote the Borel subgroup $\left\{ \left(\begin{smallmatrix} \alpha & b \\ 0 & \alpha^{-1} \end{smallmatrix}\right) \right\}$ and let the matrices

$$s_u = \begin{pmatrix} 0 & 1 \\ -1 & -u \end{pmatrix},\ u = 0, 1, \ldots, q-1 \text{ and } s_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

be a fixed set of coset representatives of $B \backslash SL_2(\mathbb{F}_q)$, which we identify with $\mathbb{P}^1(\mathbb{F}_q)$. In [8] we constructed the principal series representations of $SL_2(\mathbb{F}_q)$ by inducing characters from B, and expressing these matrices in terms of their action on $\{s_u\}$. Very briefly, if $\psi : \mathbb{F}_q^\times \to \mathbb{C}$ is a character of B, then the induced representation $\rho_\psi = \psi \uparrow B$ is given as

$$\rho_\psi(g) s_u = \psi(f(u,g)) s_{g \cdot s_u} \tag{3.8}$$

for some function $f(u,s) \in \mathbb{F}_q^\times$. In particular, $\widehat{\delta_{\mathcal{S}}}(\rho_1)$ is the adjacency matrix $\Gamma(\mathbb{P}^1(\mathbb{F}_q), \mathcal{S})$; we refer to [8] for details. If $\Gamma(PSL_2(\mathbb{F}_q), \mathcal{S})$ is a good expander, then $\Gamma(\mathbb{P}^1(\mathbb{F}_q), \mathcal{S})$ is also a good expander since its spectral gap is

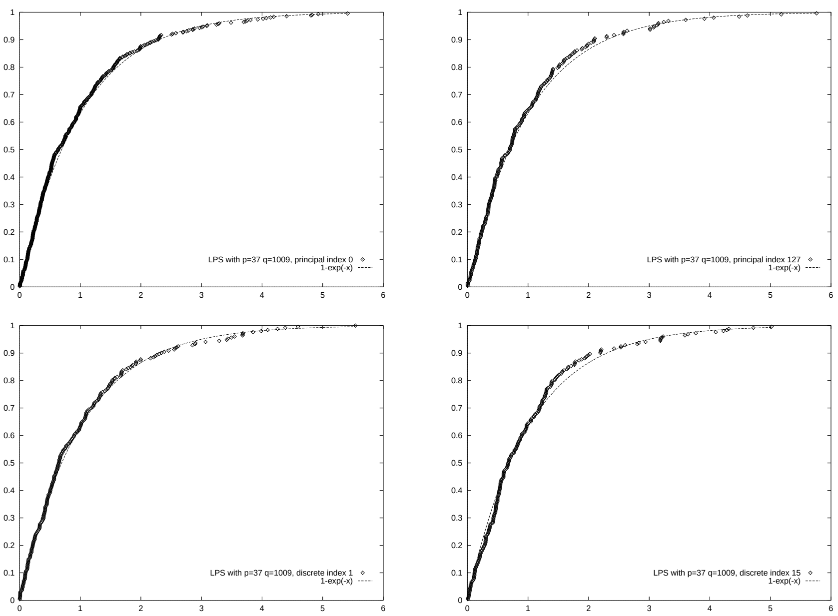

FIG. 3.5. *$P(S)$ for individual Fourier transforms, for LPS generators with $p = 37$ and $q = 1009$. Top row: two principal series representations, action on projective line on the left. Bottom row: two discrete series representations.*

at least as large. In particular, the explicit Ramanujan graphs for $PSL_2(\mathbb{F}_q)$ restrict to give Ramanujan graphs on $\mathbb{P}^1(\mathbb{F}_q)$ [18].

From a computational point of view, the discrete series representations are more expensive to construct. However, our experience has shown that the spectral properties of the principal and discrete series representations are almost identical. In Figure 3.5 the spacing distribution is shown for the 38-regular LPS graphs $X^{37,1009}$ for $PSL_2(\mathbb{F}_{1009})$ at two principal series and two discrete series representations.

4. Summary. We have computed the cdf for the eigenvalue spacings of a variety of Cayley graphs on $SL_2(\mathbb{F}_p)$. When the graphs are good expanders, the distributions are Poisson, even on the level of the individual Fourier transforms. The data suggests and supports the following conjecture.

Conjecture 1. *Let $X_p = X(PSL_2(\mathbb{F}_p), \mathcal{S})$ be a family of k-regular Cayley graphs such that $\lambda_1(\Gamma_p) \to 2\sqrt{k-1}/k$ as $p \to \infty$. Then asymptotically as $p \to \infty$, the distributions of the eigenvalue spacings for X_p and its individual Fourier transforms $\widehat{\delta}_S(\rho)$ are Poisson.*

Additionally, our randomly generated 2-generator Cayley graphs on $SL_2(\mathbb{F}_p)$ also exhibit Poisson behavior. In light of our previous experiments

[8,9] which suggest that a randomly chosen 2-generator Cayley graph on $SL_2(\mathbb{F}_p)$ is a good expander, this too suggests a relationship between expansion and Poisson spacing. In this spirit we anticipate a "central limit theorem" for random 2-generator Cayley graphs on $SL_2(\mathbb{F}_p)$.

Conjecture 2. *For $\epsilon > 0$, as $p \to \infty$ the probability of choosing a pair of generators $\mathcal{S}$ for $SL_2(\mathbb{F}_p)$ with $\|P(S) - (1 - e^{-x})\| \geq \epsilon$ goes to zero.*

Availability of Software. All of our computations were performed on HP 735/125 and DEC 3000 Model 600 Alpha workstations, using software written in the C language. The code will be available via the web page `www.cs.dartmouth.edu/~rockmore/GFT`.

REFERENCES

[1] M. Berry, *Some quantum to clasical asymptotics*, in Chaos and Quantum Physics, M.-J. Giannoni, A. Voros, and J. Zinn-Justin (eds.), Elsevier Pub., NY, 1991, pp. 252–303.

[2] P. Diaconis, *Group Representations in Probability and Statistics*, Inst. of Math. Stat., Hayward, CA, 1989.

[3] P. Diaconis and M. Shashahani, *On the eigenvalues of random matrices*, J. Appl. Probab., **31A** (1994), 49–62.

[4] M. Gutzwiller, *Quantum chaos*, Scientic American, January 1992, pp. 78–84.

[5] D. Jakobson, S. Miller, I. Rivin and Z. Rudnick, *Eigenvalue spacings for regular graphs*, IMA, this volume.

[6] R. Kannan, P. Tetali, and S. Vempala, *Simple Markov-chain algorithms for generating bipartite graphs and tournaments*, ACM Symposium on Discrete Algorithms, 1997, to appear.

[7] N. Katz and P. Sarnak *The spacing distributions between zeros of zeta functions*, preprint, 1996.

[8] J. Lafferty and D. Rockmore, *Fast Fourier analysis for SL_2 over a finite field and related numerical experiments*, Experimental Mathematics **1** (1992), pp. 115–139.

[9] J. Lafferty and D. Rockmore, *Numerical investigation of the spectrum for certain families of Cayley graphs*, in DIMACS Series in Disc. Math. and Theor. Comp. Sci., Vol. 10, J. Friedman (ed.), (1993), pp. 63–73.

[10] A. Lubotzky, *Discrete Groups, Expanding Graphs, and Invariant Measures*, Birkhäuser, Boston, 1994.

[11] A. Lubotzky, R. Phillips and P. Sarnak, *Ramanujan graphs*, Combinatorica, **8** (1988), pp. 261–277.

[12] W. Luo and P. Sarnak, *Number variance for arithmetic hyperbolic surfaces*, Comm. Math. Phys., **161** (1994), 419–432.

[13] M. L. Mehta, *Random Matrices*, Academic Press Inc., San Diego, 1991.

[14] M. Naimark and A. Stern, *Theory of Group Representations*, Springer-Verlag, NY, 1982.

[15] A. Odlyzko, *On the distribution of spacings between zeros of zeta functions*, Math. Comp., **48** (1987), pp. 273–308.

[16] Z. Rudnik and P. Sarnak, *The behavior of eigenstates of arithmetic hyperbolic manifolds*, Comm. Math. Phys., to appear.

[17] P. Sarnak, *Arithmetic chaos*, Israel Math. Conf. Proc., Vol. 8, (1995), pp. 183–236.

[18] P. Sarnak, *Some Applications of Modular Forms*, Cambridge Univ. Press, Cambridge, 1990.

[19] C. Schmit, *Quantum and classical properties of some billiards on the hyperbolic plane*, in Chaos and Quantum Physics, M.-J. Giannoni, A. Voros, and J. Zinn-Justin (eds.), Elsevier Pub., NY, 1991, pp. 333-369.

[20] A. Selberg, *On the estimation of Fourier coefficients of modular forms*, Proc. Symp. Pure Math. **8** (1965), pp. 1–15.

[21] J. P. Serre, *Linear Representations of Finite Groups*, Springer-Verlag, NY, 1986.

[22] Y. Shalom, *Expanding graphs and invariant means*, preprint, 1996.

[23] A. Silberger, *An elementary construction of the representations of* $SL(2, GF(q))$, Osaka J. Math. **6** (1969), pp. 329–338.

[24] J. Wilkinson, *The Algebraic Eigenvalue Problem*, Oxford Univ. Press, 1965.