Edge Coloring, Polyhedra and Probability

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Abstract

Edge Coloring is the following optimization problem:

> Given a graph, how many colors are required to color its edges in such a way that no two edges which share an endpoint receive the same color?

The required number of colors is called the chromatic index of G and is denoted by $\chi'(G)$. We consider the edge coloring problem in the framework of the relationship between an integer program and its linear programming relaxation. To do this we first formulate edge coloring as an integer program and let $\chi^*(G)$ be the optimum of the linear programming relaxation (called the fractional chromatic index). For any graph G one can compute $\chi^*(G)$ in polynomial time but deciding whether $\chi'(G) = \Delta$ or $\chi'(G) = \Delta + 1$ is NP-Complete. So it would be of interest to determine for which simple graphs $\chi'(G) = \lceil \chi^*(G) \rceil$ as we can compute $\chi'(G)$ for graphs in these classes.
In this thesis we show that large classes of graphs satisfy this equality. More precisely, we show that if a graph G is large enough, has large maximum degree and satisfies two technical conditions, then the equality holds. The constructive proof provides a randomized polynomial time algorithm for optimally coloring the edges of such graphs. We use a deterministic version of this algorithm to design the first algorithm that computes an optimal edge coloring of any graph in polynomial time, on average.

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This dissertation required from me a tenacity and persistence that I could have only obtained from my mother. I thank her, and also my sisters, for setting an example that I should always try to follow.

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Dedicated to Marija Janković (1904 – 1998)

Contents

Chapter 1

Introduction

1.1 The bird's eye view

Edge Coloring is the following optimization problem:

> Given a graph, how many colors are required to color its edges in such a way that no two edges which share an endpoint receive the same color?

This question is one of the oldest in graph theory. As pointed out by Tait [Tai80] in 1880, the celebrated four color conjecture is equivalent to the statement that the edges of every bridge-less, cubic planar graph can be colored using three colors (for an overview of the terminology, see section 1.3). Several years later, in 1891, Petersen [Pet91], while studying the factorization of certain polynomials, pointed out that there are bridge-less cubic graphs which are not three edge colorable (for example the infamous Petersen graph, in figure 1.1).

In 1916 König [KÏ6], while studying the factorization of the determinants of matrices, proved that for every bipartite graph G, the minimum number of colors needed to color the edges of G is equal to the maximum vertex degree in G. This theorem can be seen as a consequence of König's Bipartite Matching Min-Max theorem; König's theorem is in turn equivalent to Menger's Theorem [Men27] proved in 1927, Hall's Theorem [Hal35] proved in 1935, the Birkhoff-Von Neumann Theorem [Bir46] proved in 1946, Dilworth's Theorem [Dil50] proved in 1950, and Ford and Fulkerson's Max-Flow Min-Cut

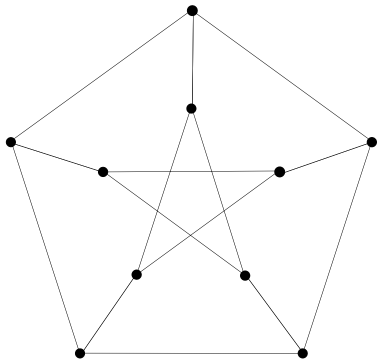

Figure 1.1: Petersen graph

Theorem [FF56] proved in 1956. (For a study of these equivalences see [FF62] and [LP86]). Thus, we see that edge coloring problem is also linked to important developments in combinatorial optimization.

Let the chromatic index of G, denoted by $\chi'(G)$ or simply χ', be the minimum number of colors required to color the edges of G. Let $\Delta(G)$ or simply Δ be the maximum vertex degree in G. It is easy to see that $\chi'(G) \geq \Delta$. Note that König's edge-coloring theorem states that $\chi'(G) = \Delta$ if G is bipartite. We remark however that not every graph is Δ edge colorable. For example, if G is an odd cycle, $\Delta(G) = 2$ colors are not enough to color the edges of G; a third color is required. In 1964, Vizing [Viz64] proved that $\Delta(G)+1$ colors are sufficient in general: in other words, every simple (without loops or multiple edges) graph G of maximum degree Δ has chromatic index $\chi' = \Delta$ or $\chi' = \Delta + 1$.

The algorithmic aspects of computing the chromatic index and an optimal edge-coloring also have a long history. A polynomial time edge-coloring algorithm for simple bipartite graphs easily follows from the original constructive proof by König. In the general case, Fournier [Fou73] applied ideas from Vizing's original proof to develop a polynomial time algorithm that colors the edges of a simple graph with $\Delta+1$ colors. Fournier's algorithm actually uses Δ colors if the vertices of maximum degree in the input graph induce an acyclic subgraph (i.e. a forest).

Given these positive algorithmic results, it is somewhat surprising that optimally coloring the edges of an arbitrary simple graph is hard. In 1981, Holyer showed that the question of deciding whether $\chi'(G) = \Delta$ or $\chi'(G) = \Delta + 1$

is $\mathcal{NP}$-complete.

In order to further motivate our interest in the edge coloring problem and provide insight into the heart of its difficulties, we introduce, as Lovasz and Plummer did in [LP86], a useful framework, originally introduced by Stahl [Sta79] and Seymour [Sey79]. We first formulate the edge-coloring problem as an integer program. To this end, let $\mathcal{M}(G)$ (or simply $\mathcal{M}$) be the set of all matchings of G and let $\mathcal{M}_e(G)$ (or simply $\mathcal{M}_e$) be the set of all matchings of G that contain the edge $e \in E$. Then

$$\chi'(G) = \min \sum_{M \in \mathcal{M}} y_M \tag{1.1}$$

subject to:

(i) $\sum_{M \in \mathcal{M}_e} y_M = 1$, for all $e \in E$,

(ii) $y_M \in \{0, 1\}$, for all $M \in \mathcal{M}$,

By removing the integrality condition from **(ii)**, we obtain the linear program

$$\min \sum_{M \in \mathcal{M}} y_M \tag{1.2}$$

subject to:

(i) $\sum_{M \in \mathcal{M}_e} y_M = 1$, for all $e \in E$,

(ii) $y_M \geq 0$, for all $M \in \mathcal{M}$,

We call the solution to this linear program the **fractional chromatic index** of G and denote it by $\chi^*(G)$, or simply χ^*. Obviously, $\chi' \geq \chi^*$ and it is easy to check that $\chi^* \geq \Delta$. The important observation is that χ^* can be computed in polynomial time, with the ellipsoid algorithm (see the discussion in section 1.3).

There is another, more combinatorial approach to understanding the fractional chromatic index. Edmond's characterization of the matching polytope ([Edm65a]) yields a formula for $\chi^*(G)$ (see [Sey79, Sta79]):

$$\chi^*(G) = \max\{\Delta(G), \max_{H\in\mathcal{H}} \frac{2|E(H)|}{|H|-1}\} \tag{1.3}$$

where $\mathcal{H}$ is the set of all subgraphs of G with an odd number of 3 or more vertices (in order to avoid cumbersome notation, we slightly abuse notation by denoting by H the set of vertices of the subgraph H). A graph $H \in \mathcal{H}$ is called **overfull** if $|E(H)| > \Delta(H)\frac{|H|-1}{2}$; we note that $|H|$ must be odd and at least 3. Since $|H|$ is odd, in any edge coloring of H, at most $\frac{|H|-1}{2}$ edges in $E(H)$ may have the same color, so at most $\Delta(H)\frac{|H|-1}{2}$ edges of H can be colored with $\Delta(H)$ colors. Consequently, $\chi'(H) = \Delta(H) + 1$. If a simple graph G contains an overfull subgraph H of maximum degree $\Delta(H) = \Delta(G) = \Delta$, it follows that $\chi'(G) = \Delta + 1$. Equivalently, by (1.3) $\chi'(G) \geq \chi*(G) > \Delta$ so $\chi'(G) = \Delta + 1$. An important paper by Padberg and Rao [PR82] on computing minimum odd-cuts gives a tool for determining an overfull subgraph of G of maximum degree Δ in time $O(n^4)$. Using their algorithm, one can compute $\chi^*(G)$ in polynomial time in a purely combinatorial way.

A study of the relationship between χ' and χ^* thus proves to be a valuable one. It is of interest to determine for what classes of simple graphs $\chi'(G) = \lceil\chi^*(G)\rceil$ as we can compute $\chi'(G)$ for graphs in these classes. More generally, the relationship between χ' and χ^* is an instance of a fundamental question in combinatorial optimization: what is the relationship between an Integer Program and its Linear Programming relaxation?

Many of the seminal results mentioned above describe classes of graphs for which $\chi'(G) = \lceil\chi^*(G)\rceil$. König's theorem basically says that for bipartite graphs $\chi' = \chi^* = \Delta$. An equivalent statement of the four color theorem, proved by Appel and Haken [AH76] in 1976, is that for planar cubic graphs $\chi' = \lceil\chi^*\rceil$. In a recent development, Robertson, Seymour and Thomas [RST97] have generalized this result to cubic graphs with no Petersen graph minor by proving Tutte's longstanding edge-coloring conjecture:

Theorem 1 (Tutte's conjecture) *If a 2-edge connected cubic graph G contains no Petersen graph as a minor then G is 3 edge-colorable.*

In the same vein, Reed [Ree95] has shown that a similar relationship holds for a generalization of bipartite graphs. We call a graph G **near-bipartite** if for some vertex x of V, the graph $G-x$ is bipartite. Reed [Ree95] proved that if G is near-bipartite then $\chi'(G) = \lceil \chi^*(G) \rceil$. He also presented a polynomial-time algorithm for optimally edge-coloring near-bipartite graphs. As we will require this result, we present the algorithm in detail in Appendix A.

A number of conjectures have been proposed regarding the relationship between χ' and χ^*. Goldberg [Gol73, Gol84], Andersen [And77] and Seymour [Sey79] independently proposed to generalize Vizing's theorem to multigraphs (graphs with multiple edges). Specifically, we have the following conjecture:

Conjecture 1 (Goldberg-Seymour conjecture)

$$\chi' \leq \max\{\Delta + 1, \lceil \chi^* \rceil\}$$

Seymour also suggested a weaker conjecture equivalent to the statement $\chi' \leq \max\{\Delta, \chi^*\} + 1$. The Goldberg-Seymour conjecture has been shown to hold for graphs with maximum degree up to 11 (see [And77, Gol77, Gol84, NK85, NK90]). Marcotte [Mar90] proved that the conjecture is true for any graph that does not contain K_5^- as a minor (K_5^- is obtained from K_5 by deleting an edge). Planholt and Tipnis [PT91] have verified the conjecture when Δ is very large, relative to $|V|$ (Δ must be on the order of $\mu(G)|V(G)|$ where $\mu(G)$ is the multiplicity of G). Finally, Kahn [Kah96] has shown that the conjecture is true asymptotically.

We will not have much to say about the Goldberg-Seymour conjecture in this thesis. Of more importance to us is the following conjecture proposed by Hilton [CH86], which does imply the Goldberg-Seymour conjecture for all graphs for which $|V(G)| < 3\Delta(G)$:

Conjecture 2 (Hilton's overfull subgraph conjecture) *If $G = (V, E)$ is simple and $\Delta(G) > \frac{1}{3}|V|$ then $\chi'(G) = \lceil \chi^*(G) \rceil$.*

The graph G obtained from Petersen's graph by removing a vertex has chromatic index $\chi' > 3$ but contains no overfull subgraph H such that $\Delta(H) = \Delta(G)$. So, we clearly cannot decrease $\frac{1}{3}$ in the above conjecture (except, perhaps, if we are are willing to forbid certain subgraphs). The conjecture is true for all G such that $\Delta(G) \geq |V(G)| - 3$, as proven by Plantholt [Pla81, Pla83] and by Chetwynd and Hilton [CH84b, CH84a, CH89b]. Our interest in Hilton's conjecture is further motivated by the observation that the proportion of graphs on n vertices not satisfying the conditions of Hilton's conjecture is very, very small (at most e^{-cn^2} for some $c > \frac{1}{40}$). This minuscule probability motivates us to to constructively prove Hilton's conjecture, or at least a theorem with similar conditions as Hilton's conjecture, and use the construction to design an algorithm that efficiently and optimally colors the edges of a huge proportion of all graphs.

The following special case of Hilton's conjecture has been "going around" since the early 1950s, according to G. A. Dirac (see [Hil89]):

Conjecture 3 *If $G = (V, E)$ is a Δ-regular simple graph with $2k$ vertices for some $k \leq \Delta$ then G is 1-factorizable (Δ edge colorable).*

An interesting consequence of this conjecture is that for any regular graph G either G or its complement has a 1-factorization. Chetwynd and Hilton [CH89a] have proved the conjecture if $\Delta \geq \frac{(\sqrt{7}-1)}{2}|V|$. Furthermore Chetwynd and Hilton [CH85] note that R. Häggkvist has announced that for any $\epsilon > 0$ there exists $N > 0$ so that every Δ-regular graph G is 1-factorizable if G has an even number of vertices greater than N and $\Delta \geq (\frac{1}{2} + \epsilon)|V|$. We stress, however, that Conjecture 3 still remains unresolved.

A number of additional conjectures regarding the edge-coloring properties of graphs with high maximum degree have been proposed and can be found in [JT95]. Many of them, however, would follow from an affirmative answer to Hilton's conjecture.

1.2 The central results

The following theorem is a central result of this thesis:

Theorem 2 (main theorem) *There exists Δ_0 such that for all simple graphs $G = (V, E)$ with maximum degree $\Delta \geq \Delta_0$ and $n = |V| \leq 6\Delta$, one of the following is true:*

(i) *G contains a subgraph H of minimum degree $\delta(H) \geq \Delta - \Delta^{79/80}$ and either:*

H is bipartite, or

$|V - H| > \Delta - 8\Delta^{159/160}$

(ii) *G contains an overfull subgraph H of maximum degree Δ,*

(iii) *G is Δ edge colorable.*

Furthermore, there is a $O(n^4)$ randomized procedure and a $O(2^n)$ deterministic procedure, both of which will output either a Δ edge coloring of G or a subgraph H of G that satisfies one of (i) or (ii).

The lower bound on the maximum vertex degree Δ_0 satisfies a number of inequalities that appear throughout this paper. A corollary of this theorem is an asymptotic version of Hilton's conjecture with $\frac{1}{3}$ replaced by $\frac{1}{2-\epsilon}$ (for any $\epsilon > 0$):

Corollary 3 *For every $\epsilon > 0$, there is some Δ_1 such that for all simple graphs $G = (V, E)$ with maximum degree $\Delta \geq \Delta_1$, if $\Delta \geq \frac{1}{2-\epsilon}|V|$ then*

$$\chi'(G) = \lceil \chi^*(G) \rceil$$

We note that when restricted to regular graphs, this corollary is equivalent to the result announced by R. Häggkvist: for any $\epsilon > 0$ there exists $N > 0$ so that every Δ-regular graph G is 1-factorizable if G has an even number of vertices greater than N and $\Delta \geq (\frac{1}{2} + \epsilon)|V|$.

The main application of our theorem (more precisely, of our deterministic edge-coloring algorithm) is the other central result in this thesis. We present a deterministic algorithm *FAST COLOUR* that computes an optimal edge coloring of any simple graph in polynomial time on average, assuming a uniform distribution of the input graphs.

1.3 Preliminaries

In this section, we give a sampling of relevant results from graph theory, algorithms and complexity theory, polyhedral combinatorics and probabilistic methods. We choose to prove in detail the results we use extensively in this thesis.

1.3.1 Graph theory

A graph $G = (V, E)$ is defined by a set of vertices $V = V(G)$ and a multiset of edges $E = E(G)$ between pairs of different vertices. Note that we allow multiple (or parallel) edges but do not allow loops (e.g. (v, v) for some $v \in V$). The **multiplicity** of an edge $(u, v) \in E$ is the number of occurrences of (u, v) in E and is denoted by $\mu(u, v)$. The multiplicity of G is defined as $\mu = \max\{\mu(u, v) : (u, v) \in E\}$. If $\mu \leq 1$, we call G **simple**.

We denote by $E(H)$ the set of edges induced by a subset H of V, i.e. all edges in E with both endpoints in H. We often will abuse notation and denote by H the subgraph of G defined by $(H, E(H))$. We define $E(X, Y) = \{(x, y) \in E(G) : x \in X \text{ and } y \in Y\}$ for disjoint subsets X and Y of $V(G)$, and we call (X, Y) the bipartite subgraph with vertex set $X \cup Y$ and edge set $E(X, Y)$. (We recall that G is **bipartite** if its vertices can be partitioned into sets A and B such that A and B induce empty subgraphs.)

Two vertices are adjacent if they are endpoints of some edge in E, and an edge is incident to a vertex v if v is an endpoint of the edge. The degree of a vertex v in the graph G is the number of edges of G incident to v and is denoted $d^G(v)$, or simply $d(v)$. We also use $d_S^G(v) = |E(\{v\}, S - \{v\})|$, or simply $d_S(v)$. The maximum vertex degree in G is denoted by $\Delta(G)$ or simply Δ and the minimum vertex degree in G is denoted by $\delta(G)$ or simply δ. We emphasize that if $H \subset V$, then $\Delta(H) = \Delta(H, E(H))$ and $\delta(H) = \delta(H, E(H))$. G is r-**regular** if $\Delta = \delta = r$.

A **matching** in a graph $G = (V, E)$ is a set of edges no two of which share an endpoint. A **k edge coloring** of a graph is a partition of its edges into k matchings. The **chromatic index** of G, denoted $\chi'(G)$ or simply χ', is the least k for which a k edge coloring of G exists. It is easy to see that $\chi'(G) \geq \Delta$. For a deeper, although somewhat outdated, treatment of edge coloring, see [FW77]. Given a matching M of G, a path P is M-alternating if its edges are alternately in and out of M. If, in addition, the first edge and the last edge of P do not belong to M, the path is M-augmenting (see figure 1.2); we note that a matching of cardinality $|M| + 1$ is obtained by removing all edges of $M \cap P$ from M and adding the remaining edges of P to M. Given matchings M and N, a path P is MN-alternating if its edges are alternating between edges in M and edges in N.

G' is a **reduction** of G if $\Delta(G') = \Delta(G) - l$ and there exists a set of matchings $\{M_1, M_2, ..., M_l\}$ such that $G' = G - M_1 - M_2 - ... - M_l$. We

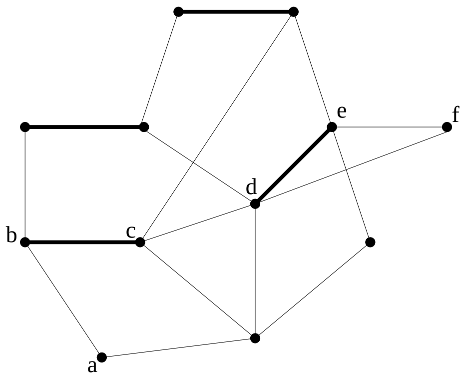

Figure 1.2: Matching M (bold edges) and an M-augmenting path

remark that given disjoint matchings $M_1, ..., M_l$ in a graph $G = G_0$ and setting $G_i = G_{i-1} - M_i$ we have that each G_i is a reduction of G if and only if for each i, every vertex of maximum degree in G_{i-1} is the endpoint of some edge of M_i. The observation critical to our problem is that if G' is k edge colorable then G is $k + l$ colorable; in particular, if G' is $\Delta(G')$ edge colorable, then G is Δ edge colorable.

For the additional graph theoretic concepts that we use, look in [BM76]. In the remainder of this discussion, we give the details of Fournier's [Fou73] edge coloring algorithm and in the process we prove Vizing's adjacency lemma [Viz64]. We will repeatedly make use of these throughout the thesis.

Fournier's algorithm and Vizing's adjacency lemma

In order to color the edges of a simple graph $G = (V, E)$ of maximum degree Δ with $k > \Delta$ colors, we iteratively color an additional edge of G until all edges of G are colored. If $k = \Delta$, since not all graphs are Δ edge colorable, we will give a sufficient condition which, if satisfied, allows us to extend a Δ edge coloring of $G - (u, v)$ to a coloring of G. We extend a k edge coloring of $G - (v, w)$, where (v, w) is an edge of G, to a k edge coloring of G as follows:

1. Let α and β be some colors among the k colors missing at v and w, respectively. Those colors must exist since both v and w are incident to at most $\Delta - 1$ colored edges. If $\alpha = \beta$ we easily color (v, w) with color α. If v and w belong to separate components of the subgraph $\alpha\beta$, defined by all α- and β-colored edges of $G - (v, w)$, we interchange the colors α and β on the connected (path) component starting at v

in $\alpha\beta$ (so that v misses β) and we color (v, w) with color β.

2. If we are not successful in step **1.**, then v and w are the endpoints of a connected (path) component of $\alpha\beta$. Furthermore, we can assume that for any color γ missing at w, v and w are the endpoints of a connected (path) component of $\alpha\gamma$ (otherwise, we can color (v, w) in step **1.** by replacing β with γ). We now construct a *recoloring sequence* of distinct vertices $w_1 = w, w_2, ..., w_l$ adjacent to v defined so that:

(i) there are distinct color classes $\beta_1 = \beta, ..., \beta_{l-1}$ with β_i missing at w_i and (v, w_{i+1}) is colored β_i,

(ii) for every $i = 1, ..., l-1$ and for every color γ missed at w_i, v and w_i are the endpoints of a connected (path) component of $\alpha\gamma$ (note that (v, w_{i+1}) belongs to the connected (path) component of $\alpha\beta_i$),

(iii) there is some color β_* missing at w_l, where either:

a. β_* misses v, or

b. w_l and v belong to separate connected components of $\alpha\beta_*$, or

c. $\beta_* = \beta_i$ for some $i < l-1$.

Given such a recoloring sequence, we construct a k-edge coloring as follows. We first move each color β_i (for $i = 1, ..., l-1$) from (v, w_{i+1}) to (v, w_i). This leaves (v, w_l) uncolored. In case **a.**, we color (v, w_l) with color β_*. In case **b.**, we interchange the colors α and β_* on the connected (path) component starting with v (so that v misses B^*) and we color (v, w_l) with color β_*. In case **c.**, we note v and w_l are in separate components of $\alpha\beta_i$: the connected component starting from v with the edge (v, w_i) ends in w_{i+1}. We interchange the colors α and β_i on the connected (path) component starting at w_l (so that v misses α) and we color (v, w_l) with color α.

We construct a recoloring sequence iteratively and we start each iteration i by picking some β_i missed at vertex w_i. Since the colors $\beta_1, ..., \beta_l$ are all distinct and different from α, l must be smaller than the total number of available colors k. The important observation, then, is that a recoloring sequence can be constructed if and only if at every iteration i there is some β_i missing at w_i. Clearly, if $k \geq \Delta + 1$, there always is such a color so that G is always k edge colorable (Vizing's theorem).

If $k = \Delta$ however, some vertex w_l of degree Δ may have no missing colors, and a recoloring sequence starting with vertex w_1 missing color β_1 may be impossible to construct. In this case, we call the sequence of edges $(v, w_2), ..., (v, w_l)$ a *fan sequence* induced by β_1. While there may be different fan sequences induced by β_1, we observe that no two fan sequences induced by different colors missed at w_1 can share an edge. This is because if one traverses the fan sequence backwards from (v, w_i) colored with color β_{i-1}, then the sequence of edges $(v, w_{i-1}), (v, w_{i-2}), ..., (v, w_1)$, and consequently $\beta_{i-2}, ..., \beta_1$, is uniquely determined. So for every color missing at w_1 for which we cannot construct a recoloring sequence, there exists at least one edge (v, w_l) such that w_l has maximum degree and such that (v, w_l) does not belong to any fan sequence induced by another color missed at w_1 .

This argument implies that we are able to Δ edge color G if $G - (v, w)$ has a Δ edge coloring such that the number of colors missing at w is greater than the number of neighbors of v with maximum degree. This observation is exactly Vizing's adjacency lemma:

Lemma 4 (Vizing's Adjacency Lemma) *Let $G = (V, E)$ be a simple graph of maximum degree Δ such that $G - (v, w)$ is Δ edge colorable, for some edge $(v, w) \in E$. If*

$$d(w) + |\{x \in V : (x, v) \in E \text{ and } d(x) = \Delta\}| < \Delta + 1$$

then G is also Δ edge colorable.

The algorithm we described extends a k edge coloring of $G - (v, w)$ to a k edge coloring of G in $O(n^3)$ time. We actually do a bit of extra work: $O(n^2)$ time suffices. Thus, a graph can be k edge colored using Fournier's algorithm in time $O(n^4)$. We will also use the multigraph version of Vizing's adjacency lemma, along with the corresponding $O(n^4)$ edge coloring algorithm:

Lemma 5 (Vizing's Adjacency Lemma, for multigraphs) *Let $G = (V, E)$ be a multigraph of maximum degree Δ and multiplicity μ such that $G - (v, w)$ is Δ edge colorable, for some edge $(v, w) \in E$. If*

$$d(w) + \sum_{x \in X} \mu(v, x) < \Delta + 1$$

where $X = \{x \in N(v) : d(x) > \Delta - \mu(v, x)\}$, then G is also Δ edge colorable.

1.3.2 Algorithms and complexity

We give some standard terminology for classifying problems by their underlying complexity (much more detail can be found in [GJ79, Pap94, Sip96]). We consider only **decision problems**, i.e. problems that can be phrased as a question with a positive or negative answer. For example, the perfect matching problem can be phrased as: "Does G contain a perfect matching?", while the the edge coloring problem for simple graphs can be stated as: "Does G permit a Δ edge coloring?". An optimization problem can be reduced to a decision problem using binary search. For example, finding a maximum matching in a graph G can be reduced to the problem: "Does G contain a matching of size k?".

The class of problems solvable in time that is bounded by a polynomial with respect to the input size is denoted by $\mathcal{P}$. A problem is in the set $\mathcal{NP}$ if for every input that has a positive answer there is a certificate from which the correctness of the answer can be derived in polynomial time (more concisely, a em good certificate). The perfect matching problem is in $\mathcal{NP}$ since a perfect matching is a good certificate. The edge coloring problem also belongs to $\mathcal{NP}$, since a Δ edge coloring of an input graph G is good certificate. Most other combinatorial optimization problems are also in $\mathcal{NP}$, which trivially contains $\mathcal{P}$. While it is an open question whether the containment is proper, it is widely believed that it is.

$\mathcal{NP}$-complete problems are the hardest problems in $\mathcal{NP}$: if one $\mathcal{NP}$-complete can be solved in polynomial time, then then every $\mathcal{NP}$-complete problem can be solved in polynomial time. The edge coloring problem is $\mathcal{NP}$-complete as shown by Holyer [Hol81] while the maximum matching problem (and consequently the perfect matching problem) is not (assuming $\mathcal{P} \neq \mathcal{NP}$) by Edmonds' polynomial time maximum matching algorithm [Edm65b].

A problem is in co-$\mathcal{NP}$ if, for every input that has a negative answer, there is a good certificate (i.e. a certificate from which the correctness of the negative answer can be deduced in polynomial time). A problem in $\mathcal{P}$ is trivially in co-$\mathcal{NP}$ because we can check a negative answer by simply solving the problem using a polynomial time algorithm. More generally, a problem in $\mathcal{NP}\cap$ co-$\mathcal{NP}$ has good certificates for both positive and negative answers. The perfect matching problem is in co-$\mathcal{NP}$ because of the following seminal result:

Theorem 6 (Tutte's theorem) *[Tut47] A graph $G = (V, E)$ does not* *contain a perfect matching if and only if there exists* $S \subseteq V$ *such that* $o(G - S) > |S|$

where $\mathrm{o}(G - S)$ is the number of odd connected components of $G - S$. Because Tutte's theorem gives good certificates for both positive and negative answers, it is called a *good characterization.*

It is unknown whether $\mathcal{NP} =$ co-$\mathcal{NP}$. If the answer is negative (as it is believed to be), no $\mathcal{NP}$-complete problem can have a good characterization. For example, an overfull subgraph of maximum degree $\Delta(G)$ is a certificate for a graph not being Δ edge colorable, but there exist graphs that are not Δ edge colorable which have no such overfull subgraph. So, this does not put edge coloring into co- $\mathcal{NP}$. However, an overfull subgraph of maximum degree $\Delta(G)$ is a good certificate for many important classes of graphs. This thesis explores, among other things, the classes of graphs that accept an overfull subgraph as a good certificate for a negative answer to an edge coloring problem.

We remark that in the above discussion, strict guarantees on performance, such as worst-case running time and determinism, are assumed. Other computing performance paradigms, with more tolerant requirements, have been proposed and used. In 1.3.4 we will discuss the uses of randomization in the design and analysis of algorithms.

1.3.3 Polyhedral combinatorics

We discuss here some of the polyhedral combinatorics issues that this thesis addresses. For an in-depth overview of polyhedral combinatorics, see [Sch86, CCPS98].

We recall from the introductory section 1.1, the definition of the fractional chromatic index using the following linear program:

$$\chi^*(G) = \min \sum_{M \in \mathcal{M}} y_M \tag{1.4}$$

subject to:

(i) $\sum_{M \in \mathcal{M}_e} y_M = 1$, for all $e \in E$,

(ii) $y_M \geq 0$, for all $M \in \mathcal{M}$,

where $\mathcal{M}(G)$ (or simply $\mathcal{M}$) is the set of all matchings of G and $\mathcal{M}_e(G)$ (or simply $\mathcal{M}_e$) is the set of all matchings of G that contain the edge $e \in E(G)$. The dual of this LP is given by:

$$\chi^*(G) = \max \sum_{e \in E} x_e \tag{1.5}$$

subject to:

(i) $\sum_{e \in M} x_e \leq 1$, for all $M \in \mathcal{M}$,

(ii) $x_e \geq 0$, for all $e \in E$

Note that the dual LP may have exponentially many constraints (one for each matching). However, a candidate solution to the dual can be seen as an assignment of weights to the edges and a violated constraint (if any) can be found by solving a maximum weighted matching problem. It follows that $\chi^*(G)$ can be computed in polynomial time using the ellipsoid method (see [Kha79]).

For our purposes, the heavy handed ellipsoid method is not really necessary. We describe a strictly combinatorial approach for simple graphs that allows us to compute $\chi^*(G)$ in polynomial time.

In order to obtain a formula for the fractional chromatic index, we use Edmonds' characterization of the matching polytope [Edm65a]: given a graph G, the matching polytope $M(G)$ is the set of convex combinations of characteristic vectors (in $\{0,1\}^{|E|}$) of the matchings of G. Edmonds has shown that $M(G)$ is defined by:

(i) $x(\delta(v)) \leq 1$, for all $v \in V$,

(ii) $x(E(H)) \leq \frac{1}{2}(|H| - 1)$, for all $H \subset V$, $|H| \geq 3$ and odd,

(iii) $x_e \geq 0$, for all $e \in E$.

where x is a vector in $R^{|E|}$, $\delta(v)$ is the set of edges incident to v and $E(H)$ is the set of edges induced by vertices in H. Edmonds' characterization of the matching polytope yields a formula for χ^* (see [Sey79], [LP86]):

$$\chi^*(G) = \max\{\Delta(G), \max_{H\in\mathcal{H}} \frac{2|E(H)|}{|H|-1}\}$$

where $\mathcal{H}$ is the set of all subgraphs of G with an odd number of 3 or more vertices.

Proof: Let $i_M \in \{0,1\}^{|E|}$ be the characteristic vector of matching M, for every $M \in \mathcal{M}$, and let i_E be the vector of all ones. We observe that $\sum_{M\in\mathcal{M}} w_M = \alpha$ if and only $\sum_{M\in\mathcal{M}} \frac{w_M}{\alpha} i_M$ is a convex combination of matchings. Further $\sum_{M\in\mathcal{M}_e} w_M = 1$ is equivalent to $\sum_{M\in\mathcal{M}_e} \frac{w_M}{\alpha} = \frac{1}{\alpha}$. Thus we obtain the following fact: there exists $\{w_M\}_{M\in\mathcal{M}}$ satisfying the linear program 1.4 if and only if $\frac{1}{\alpha} i_E$ is in the matching polytope.

Using Edmonds' characterization of the matching polytope, we see that G has a fractional coloring $\{w_M\}_{M\in\mathcal{M}}$ with $\sum_{M\in\mathcal{M}} w_M = \alpha$ iff

$$\sum_{u\in N(v)} \frac{1}{\alpha} \leq 1 \tag{1.6}$$

for all $v \in V$ and

$$\sum_{v\in H} \frac{1}{\alpha} \leq 1 \tag{1.7}$$

for all $H \subset V$ such that $|H| = 2k+1$ for some integer $k \geq 1$. In other words, $\alpha \geq \Delta$ and $\alpha \geq \max_{H\in\mathcal{H}} \frac{2|E(H)|}{|H|-1}$. So the fractional chromatic index of G is the minimum α satisfying these two inequalities, i.e. it is $\max\{\Delta, \max_{H\in\mathcal{H}} \frac{2|E(H)|}{|H|-1}\}$. $\square$

In the case of a simple graph G, we can reduce the problem of computing $\chi^*(G)$ to the problem of determining whether G contains an overfull subgraph H of maximum degree Δ using binary search. (We recall that a graph H is overfull if $|E(H)| > \Delta(H)\frac{|H|-1}{2}$.) If G is Δ-regular, this is equivalent to determining if $V(G)$ has a partition (V_1, V_2) so that $|V_1|$ and $|V_2|$ are both odd (an **odd cutset**) and $|E(V_1, V_2)| < \Delta$. More generally, to determine if G has an overfull subraph of maximum degree Δ, we need to check whether an auxiliary graph contains an odd cutset with fewer than Δ edges. We obtain this auxiliary graph G^* by adding an extra vertex v^* to

V and additional edges to E that are incident to v^* so that each vertex in V has degree Δ in G^*. If $|V(G^*)|$ is even, we label all vertices of G^* **odd**, otherwise we label v^* **even** and the remaining vertices odd; note that the number of odd-labeled vertices in G^* is even. We call a partition (V_1, V_2) of $V(G^*)$ and odd cutset if V_1 and V_2 contain odd numbers of odd-labeled vertices. We observe that if H is a subset of $V(G^*) - v^*$ of odd cardinality greater than 1, then H induces an overfull subgraph in G of maximum degree Δ if and only if $|E(H, V(G^*) - H)|$ is less than Δ. So, we will be done if we can find the minimum odd cutset in G^*, i.e. the odd cutset (V_1, V_2) minimizing $|E(V_1, V_2)|$. Padberg and Rao provide an algorithm to compute, in $O(n^4)$ time, the minimum odd cutset, by modifying Gomory and Hu's [GH61] algorithm for finding the minimum cutset between a set of terminal nodes (odd-labeled vertices in our terminology).

For completeness, we summarize Padberg and Rao's recursive procedure to determine the minimum odd cutset of G^*. We first compute the minimum cutset (V_1, V_2) of G^* using a standard maximum flow - minimum cut network flow algorithm. If (V_1, V_2) is an odd cutset, we are done. Otherwise, it can be shown that there is a minimum odd cutset (V_1', V_2') of G^* such that either $V_1' \subset V_1$ or $V_2' \subset V_2$. So, we only need to recursively determine the minimum odd cutset of the subgraphs of G^* induced by $V_1 + v_2$ and $V_2 + v_1$, where v_1 and v_2 are vertices obtained by identifying all vertices in V_1 and V_2, respectively (we let v_1 and v_2 be even-labeled). Since the recursion tree contains at most $n-1$ vertices, and at each node of the tree, we only need to solve a network flow problem (in time $O(n^3)$, see [LP86]), the total running time is $O(n^4)$. We note that the maximum flow - minimum cut problem is equivalent to the problem of finding a maximum matching in a bipartite graph. This problem in turn is the core of the algorithm for edge coloring bipartite graphs (see e.g. [LP86]). Thus, in a certain sense, Padberg and Rao's algorithm computes $\chi^*(G)$ by repeatedly computing χ' for some bipartite graph.

1.3.4 Probability

A finite probability space consists of a finite set Ω (the domain) and a function $\Pr : \Omega \to [0,1]$ (the probability distribution), such that $\sum_{x \in \Omega} \Pr(x) = 1$. A probability space represents a random experiment where we choose a member of Ω at random and $\Pr(x)$ is the probability that x is chosen. For any $X \subset \Omega$, we define $\Pr(X) = \sum_{x \in X} \Pr(X)$, the probability that a member of X is chosen.

The most common probability distribution, and the only one we will be using in this thesis, is the uniform probability distribution which is defined as $\Pr(x) = 1/|\Omega|$: a uniformly chosen member of Ω is a random choice where each member is equally likely.

A random variable X is a variable defined as a function of the domain of a probability space. A random event is a random variable whose value can be true or false. The subadditivity of probabilities property states that for events $E_1, ..., E_n$,

$$\Pr(E_1 \vee ... \vee E_n) \leq \sum_{i=1}^{n} \Pr(E_i)$$

Two random events X and Y are independent if $\Pr(A \wedge B) = \Pr(A) \times \Pr(B)$. The random events $X_1, ..., X_n$ are mutually independent if $\Pr(X_{i_1} \wedge ... \wedge X_{i_k}) = \Pr(X_{i_1}) \times ... \times \Pr(X_{i_k})$ for every subset $\{i_1, i_2, ..., i_k\}$ of $\{1, 2, ..., n\}$.

The expected value (or simply the mean) of a random variable X is defined as $\mathbf{Exp}(X) = \sum_{s \in \Omega} X(s) \Pr(s)$. In order to show concentration of a random variable around its mean we will use:

Theorem 7 (The Chernoff Bound) *Let $X_1, ..., X_i, ..., X_n$ be mutually independent random variables with $\Pr(X_i = +1) = \Pr(X_i = -1) = \frac{1}{2}$ and let $S_n = X_1 + ... + X_n$. For any $a > 0$,*

$$Pr(|S_n| > a) < 2e^{\frac{-a^2}{2n}}$$

Finally, we will use the standard notation $G_{n,p}$ to denote the random graph with vertex set $V_n = \{1, 2, ..., n\}$ in which each one of the $\binom{n}{2}$ possible edges occurs independently with probability p.

We finish now the discussion on algorithms and complexity started earlier. Once we introduce probability in computation, we have at our disposal not only powerful probabilistic tools, but also the flexibility in setting more tolerant performance guarantees. By accepting that an algorithm may fail with small probability (failure can mean returning an incorrect answer or not stopping within a set time), we hope that we can show that larger classes of problems can be computed efficiently.

One approach to using probability in computation is to allow a coin to be flipped at each step of an algorithm and choose one of two possible steps depending on the outcome of the flip. The class $\mathcal{RP}$ includes all problems that, with a small probability of failure (returning NO if the correct answer is YES), can be solved in polynomial time with such a procedure. (Precise definitions of $\mathcal{RP}$ and other complexity classes based on randomized algorithms can be found in [MR95].) It is easy to see that $\mathcal{RP} \subseteq \mathcal{NP}$ and it is not known whether the containment is strict. It is believed however that $\mathcal{RP} \neq \mathcal{NP}$ so it is unlikely that there exists a randomized polynomial time algorithm for the edge coloring problem, and, for that matter, for any other $\mathcal{NP}$-complete problem.

A different approach to relaxing the performance guarantees of a deterministic algorithm is to allow the running times to be super-polynomial for only a small number of inputs. More precisely, we require the running time to be polynomial on an average input only, assuming a specified probabilistic distribution of the inputs. Expected time algorithms typically involve the application of different procedures in succession: we apply a procedure to an input if the previous procedures fail on that input. We note that an algorithm has polynomial expected time if the running time of a procedure multiplied by the probability that previous procedures failed on that input is a polynomial in the size of the input. Examples of graph problems for which expected polynomial time algorithms are known include vertex coloring [DF89] and the Hamilton cycle problem [BFF87]. A recent survey by Frieze and McDiarmid [FM97] describes the state of the art in the area of average case analysis of algorithms.

We lastly discuss two results directly relevant to our objectives. Erdös and Wilson [EW77] showed that the proportion of labelled graphs on n vertices with more than one vertex of maximum degree is at most $O(n\log n)^{-\frac{1}{2}}$. It follows from Fournier's algorithm that a random graph $G_{n,\frac{1}{2}}$ can be colored with Δ colors with probability of failure at most $O(n\log n)^{-\frac{1}{2}}$. In 1986, Frieze, Jackson, McDiarmid and Reed [FJMR88] presented a $O(n^4)$ time algorithm that optimally colors the edges of $G_{n,\frac{1}{2}}$ with probability of failure $e^{-cn\log n}$ for any $c < \frac{1}{8}$. They observed that if there is a procedure that optimally edge-colors all graphs of order n in worst-case time $e^{\alpha n\log n}$ for some $\alpha < \frac{1}{8}$, it could be the second step of a polynomial expected time algorithm, with their algorithm being the first step. Their algorithm is in fact the first in the sequence of procedures that are part of our algorithm *FAST COLOUR* that optimally colors the edges of a simple graph in polynomial time on average.

Chapter 2

The results

2.1 Graphs of high degree

Our main result is a proof of a weakening of Hilton's Conjecture for graphs with sufficiently large degree. This result allows us to develop an algorithm for edge coloring which runs in polynomial time on average. The precise result we prove is:

Theorem 8 (main theorem) *There exists Δ_0 such that for all simple graphs $G = (V, E)$ with maximum degree $\Delta \geq \Delta_0$ and $n = |V| \leq 6\Delta$, one of the following is true:*

(i) *G contains a subgraph H such that $\delta(H) \geq \Delta - \Delta^{79/80}$ and either:*

H is bipartite, or

$|V - H| > \Delta - 8\Delta^{159/160}$

(ii) *G contains an overfull subgraph H of maximum degree Δ,*

(iii) *G is Δ edge colorable.*

Furthermore, there is a $O(n^4)$ randomized procedure and a $O(2^n)$ deterministic procedure, both of which will output either a Δ edge coloring of G or a subgraph H of G that satisfies (i).

The lower bound on the maximum vertex degree Δ_0 satisfies a number of inequalities that appear throughout this paper. We also remark that, in this thesis, we sacrifice sharper bounds for clarity of presentation.

Note that if we could drop condition **(i)** then the main theorem would imply Hilton's conjecture asymptotically. We do obtain, as a corollary to the main theorem, the following asymptotic version of Hilton's conjecture with $\frac{1}{3}$ replaced by $\frac{1}{2-\epsilon}$ (where $\epsilon > 0$):

Corollary 9 *For every $\epsilon > 0$, there is some Δ_1 such that for all simple graphs $G = (V, E)$ with maximum degree $\Delta \geq \Delta_1$ and $\Delta > \frac{1}{2-\epsilon}|V|$:*

$$\chi'(G) = \lceil \chi^*(G) \rceil$$

Proof: Given ϵ, let $G = (V, E)$ be a graph of maximum degree $\Delta \geq \Delta_1 = \max\{\Delta_0, 9\epsilon^{-1/160}\}$ such that $|V| \leq (2-\epsilon)\Delta$. If G contains a subgraph H which satisfies **(i)** in the main theorem, then either H is bipartite and $|V| \geq |H| \geq 2\Delta - 2\Delta^{159/160}$ or $|V| \geq |H| + |V - H| \geq 2\Delta - 9\Delta^{159/160}$. Since $|V| \leq 2\Delta - \epsilon\Delta$, it follows that $\epsilon \leq 9\Delta^{-1/160}$, a contradiction. So either G contains an overfull subgraph of maximum degree Δ or G is Δ edge colorable. $\square$

Because Δ-regular graphs with $|V| = 2k$ and $k \leq \Delta$ can not contain an overfull subgraph of maximum degree Δ, our corollary is equivalent to the result announced by R. Häggkvist: for any $\epsilon > 0$ there exists $N > 0$ so that every Δ-regular graph G is 1-factorizable if G has an even number of vertices greater than N and $\Delta \geq (\frac{1}{2} + \epsilon)|V|$.

The proof of our main theorem is long and complicated; its details comprise chapters 3 – 6 of this thesis. In this chapter, we content ourselves with sketching the proof and explaining why we need condition **(i)**. We do this in section 2.4. To ease our exposition, we first present, in sections 2.3 and 2.5, the complete proof of the following special case of the main theorem:

Theorem 10 (regular theorem) *There exists Δ_0 such that for all simple regular graphs $G = (V, E)$ of degree $\Delta \geq \Delta_0$ with $|V| = 2k$ where $k \leq \Delta$, one of the following is true:*

(i) *G contains a subgraph H such that $\delta(H) \geq \Delta - \Delta^{79/80}$ and either:*

H is bipartite, or

$|V - H| > \Delta - 2\Delta^{79/80}$

(ii) *G is Δ edge colorable.*

Furthermore, there is a $O(n^4)$ randomized procedure and a $O(2^n)$ deterministic procedure, both of which will output either a Δ edge coloring of G or a subgraph H of G that satisfies (i).

Remark: a Δ-regular graph $G = (V, E)$ with $|V| = 2k$ and $k \leq \Delta$ cannot possibly contain an overfull subgraph of maximum degree Δ.

Thus, our description of the proof of the main theorem starts in section 2.3 and occupies the remainder of the thesis. Before beginning this discussion, we digress briefly to present a deterministic edge coloring algorithm which runs in polynomial time on average.

2.2 Edge coloring quickly, on average

We present *FAST COLOUR*, a deterministic algorithm for optimally coloring the edges of any simple graph. We will prove that the expected running time of *FAST COLOUR* on an input graph with n vertices is a polynomial in n, assuming a uniform distribution of graphs with n vertices.

Our algorithm applies a number of procedures in succession. Most of the procedures we use we have already mentioned. As a last resort we use dynamic programming. We begin our discussion by presenting the dynamic programming procedures we need. We assume that the input graph $G = (V, E)$ is the random graph $G_{n,\frac{1}{2}}$ where $n = |V|$. In other words, every **labeled** graph with n vertices is equally likely. (While the uniform distribution of labeled graphs does not correspond exactly to the uniform distribution of unlabeled graphs, the result easily extends to the second distribution.)

In describing the three dynamic programming procedures we use, we will use the fact that a graph with r vertices contains fewer than $r!\frac{r}{2}$ matchings, for we can specify a matching by first specifying an order on the vertices which breaks them into pairs and then letting the matching consist of the first j of these pairs for some $j \leq \frac{r}{2}$.

We will use the following dynamic programming algorithm:

Standard dynamic programming We compute $\chi'(H)$ for each subgraph H of G, considering subgraphs with fewer edges earlier. Observe that for each $H \subseteq G$ (with $E(H) \neq \emptyset$), $\chi'(H) = 1 + min\{\chi'(H-M)|M \in \mathcal{M}$ and M is non-empty$\}$. So, we find an optimal edge coloring of H by choosing some M attaining this minimum and adding M to an optimal colouring of $H - M$.

Since G has $2^{|E(G)|}$ subgraphs H and we spend at most $O(n!\frac{n}{2})$ time computing $\chi'(H)$, we can compute an optimal edge coloring of G in $2^{|E(G)|+O(n\log n)}$ time.

In addition to this standard dynamic programming procedure, we will also use another, more restricted, version. Given a list of forbidden colors for each vertex of G, we will need to determine if G has a Δ edge coloring such that no edge incident to a vertex v is colored with a color forbidden on v, and furthermore, we need to find such a colouring if one exists. In order to describe the restricted dynamic programming procedure we define some terminology. An **i-proper matching** is a matching no edge of which is incident to a vertex on which i is forbidden. A **proper edge colouring** is one in which for each i, the edges of colour i form an i-proper matching.

Restricted dynamic programming For each subgraph H of G and each subset S of $\{1, ..., \Delta\}$ (considering subgraphs H with fewer edges and subsets S of smaller cardinality earlier), we determine if H has a proper edge coloring using the colors of S. This will be true if and only if for some j in S and some j-proper matching M, $H-M$ has a proper edge coloring using the colors in $S - j$. If such j and M exist, we find the proper edge coloring of H using the colors from S by adding M to the proper edge coloring of $H - M$ using the colors in $S - j$.

Since G has $2^{|E(G)|}$ subgraphs, since there are at most 2^n subsets of $\{1, ..., \Delta\}$ and since there are at most $O(n!\frac{n}{2})$ matchings in G, we compute an optimal edge coloring of G in $2^{|E(G)|+O(n\log n)}$ time.

We will use the restricted dynamic programming procedure if G permits a partition of its vertex set into A and B such that $|E(A, B)|$ is "small" (which we specify below) as follows:

Extended dynamic programming For each coloring of $E(A, B)$, we determine, separately for A and B, whether the coloring extends to

$E(A) + E(A, B)$ (resp. $E(B) + E(A, B)$) using the restricted dynamic programming procedure.

Note that there are at most $\Delta^{|E(A,B)|} < 2^{|E(A,B)| \log n}$ edge colorings of $E(A, B)$. The deterministic running time of this last procedure is then $2^{|E(A)|+O(|E(A,B)| \log n)} + 2^{|E(B)|+O(|E(A,B)| \log n)}$.

The *FAST COLOUR* algorithm

We first attempt the following:

1. We apply the edge-coloring procedure by Frieze, Jackson, McDiarmid and Reed [FJMR88] to optimally color the edges of G with Δ colors.

As noted in the preliminaries (section 1.3), the running time of this procedure is $O(n^4)$ and the probability of failure on an input graph G is $e^{-cn \log n}$ for some $c < \frac{1}{8}$. If we fail in step **1**, we move to the following step:

2. We apply the algorithm by Padberg and Rao [PR82] to determine whether G contains an overfull subgraph of maximum degree Δ. If G does contain such a subgraph, we apply the edge coloring algorithm by Fournier [Fou73] to optimally color the edges of G with $\Delta + 1$ colors.

The deterministic running time of step **2** is $O(n^4)$. If G fails in step **1** and does not contain an overfull subgraph of maximum degree Δ, we apply the algorithm implied by our main theorem as follows:

3. If $|V| \leq 6\Delta$, we apply our edge-coloring procedure to optimally color the edges of G with Δ colors or to find a subgraph H of G of minimum degree $\delta(H) \geq \Delta - \Delta^{79/80}$ such that either H is bipartite or $|V - H| > \Delta - 8\Delta^{159/160}$.

By the main theorem, the deterministic running time of the third step is $O(2^n)$. However, the expected running time of the third step on the random graph G is $O(2^n e^{-cn \log n})$
$= o(1)$.

If we fail to optimally color the edges of G in steps **1**, **2** and **3**, then one of the following must be true:

(a) $n = |V| > 6\Delta$,

(b) G contains a subgraph H of G of minimum degree $\delta(H) \geq \Delta - \Delta^{79/80}$ such that $|V - H| > \Delta - 8\Delta^{159/160}$.

(c) G contains a subgraph H of minimum degree $\delta(H) \geq \Delta - \Delta^{79/80}$ such that H is bipartite.

In each one of these cases we need to apply brute force dynamic programming as follows:

4. If $|V| > 6\Delta$ (case **(a)**), we just use the standard dynamic programming procedure. In cases **(b)** and **(c)**, we set $A = H$ and $B = V - H$ and we observe that $|E(A,B)| < |A|\Delta^{79/80} < 6\Delta^{159/80}$. We simply apply the extended dynamic programming procedure in those cases.

Lemma 11 *The expected running time of step* **4** *is o(1).*

Proof: In case **(a)**, $|E| < \frac{1}{12}n^2$. Thus the deterministic running time is $2^{\frac{n^2}{12}+O(n\log(n))}$ by our analysis of the standard procedure. Using the Chernoff bound (theorem 7), we show that $\Pr(|E| < \frac{1}{12}n^2) < 2^{\frac{-n^2}{9}}$. So, the expected time on the random graph G in this case is o(1).

In case **(b)**, we show that for any fixed partition $(A,B) = (H, V - H)$ of V, the expected running time on a random graph G is $o(2^{-n})$. The lemma then follows by summing the probabilities over all 2^n partitions and subadditivity. So, by our analysis of the extended procedure, and since $|E(A,B)|\log n < o(n^2)$, it suffices to show :

$$\Pr(\text{ \textbf{(b)} holds for } (A,B))(2^{|E(A)|+o(n^2)} + 2^{|E(B)|+o(n^2)}) = o(2^{-n})$$

We actually show:

$$\Pr(\text{\textbf{(b)} holds for } (A,B))2^{|E(A)|+o(n^2)} = o(2^{-n}) \tag{2.1}$$

The result follows by symmetry.

Now, there are at most $(|A||B|)^{6\Delta^{159/80}} = 2^{o(n^2)}$ possible sets of edges between A and B with less than $6\Delta^{159/80}$ edges. The probability that a particular set of edges is exactly $E(A,B)$ is $2^{-|A||B|}$. It follows that

$\Pr(|E(A,B)| < 6\Delta^{159/80}) < 2^{-|A||B|+o(n^2)}$. Since $|B| > \Delta - 8\Delta^{159/160}$ (part of condition **(b)**) and since $|E(A)| < \frac{1}{2}|A|\Delta$, equation 2.1 follows.

In case **(c)**, it suffices to show that for any fixed partition (X, Y, B) of V the expected running time on a random graph G is $o(3^{-n})$ (where X and Y are the two sides of the bipartite graph $A = H$ and $B = V - H$). If **(c)** holds for this partition (but not **(b)**), then $d_Y(x) > \Delta - \Delta^{79/80}$ for every $x \in X$ and $d_X(y) > \Delta - \Delta^{79/80}$ for every $y \in Y$. It follows that $|E(A)| \leq |X||Y| + |E(X)| + |E(Y)| < |X||Y| + 6\Delta^{159/80}$ and also $|E(A)| < |X|\Delta + 6\Delta^{159/80}$. In addition, $|Y|\Delta > |E(X,Y)| > |X|(\Delta - \Delta^{79/80})$ and, using a symmetric argument, we obtain $||Y| - |X|| < 3\Delta^{79/80}$. So, $|X|, |Y| < 3\Delta + 2\Delta^{79/80}$. Finally, since $|B| < \Delta - 8\Delta^{159/160}$ then $|E(B)| < |E(A)|$. Thus, by our analysis of the extended procedure it suffices to show:

$$\Pr(\textbf{(c)} \text{ holds for } (X,Y,B))2^{|E(A)|+o(n^2)} = o(3^{-n}) \tag{2.2}$$

Given X, there are less than $(|X||X|)^{4\Delta^{159/80}} < 2^{8n^{159/80}\log n}$ possible sets of edges in X with less than $(3\Delta + 2\Delta^{79/80})\Delta^{79/80} < 4\Delta^{159/80}$ edges. The probability that a particular set of edges in X is exactly $E(X)$ is $2^{-\binom{X}{2}}$. By a symmetric argument on Y and because X and Y are disjoint, it follows that $\Pr(|E(X)| < 4\Delta^{159/80} \text{ and } |E(Y)| < 4\Delta^{159/80}) < 2^{-\binom{X}{2}-\binom{Y}{2}+8n^{159/80}\log n} < 2^{-|X|^2+9n^{159/80}\log n}$. Since $|E(A)| < |X|\Delta + 6\Delta^{159/80}$, equation 2.2 follows, unless $|X| < \Delta + \Delta^{159/160}$ (and $|Y| < \Delta + \Delta^{159/160}$).

If $|X| < \Delta + \Delta^{159/160}$ and $|Y| < \Delta + \Delta^{159/160}$, then $(\Delta - \Delta^{79/80})^2 < |E(X,Y)| < \Delta(\Delta + \Delta^{159/160}$ which implies that there are less than $2^{o(n^2)}$ possible sets of edges between X and Y of cardinality greater than $(\Delta - \Delta^{79/80})^2$. The probability that a particular set of edges between X and Y is exactly $E(X,Y)$ is $2^{-|X||Y|}$. So, $\Pr(|E(X,Y)| > (\Delta - \Delta^{79/80})^2 < 2^{-|X||Y|+o(n^2)}$ and $\Pr(|E(X)| < 4\Delta^{159/80}$ and $|E(Y)| < 4\Delta^{159/80}$ and $|E(X,Y)| > (\Delta - \Delta^{79/80})^2) < 2^{-|X|^2-|X||Y|+o(n^2)}$, and once again equation 2.2 follows. $\square$

2.3 The regular case

We give a short and complete proof of the main theorem in the special case of Δ-regular simple graphs with $2k$ vertices for some $k \leq \Delta$. More precisely, we prove the following:

Theorem 12 (regular theorem) *There exists Δ_0 such that for all simple regular graphs $G = (V, E)$ of degree $\Delta \geq \Delta_0$ and $|V| = 2k$ for some $k \leq \Delta$, one of the following is true:*

(i) *G contains a subgraph H such that $\delta(H) \geq \Delta - \Delta^{79/80}$ and either:*

> *H is bipartite, or*
>
> $|V - H| > \Delta - 2\Delta^{79/80}$

(ii) *G is Δ edge colorable.*

Furthermore, there is a $O(n^4)$ randomized procedure and a $O(2^n)$ deterministic procedure, both of which will output either a Δ edge coloring of G or a subgraph H of G that satisfies **(i)**.

In order to prove this theorem, we present an $O(n^4)$ algorithm that attempts to color with Δ colors the edges of a simple regular graph $G = (V, E)$ of degree $\Delta \geq \Delta_0$ and $|V| = 2k$ for some $k \leq \Delta$. Along with the graph, our edge coloring algorithm requires that a special partition (B_1, B_2) of V be given as part of the input. This partition has the property that certain sets of vertices split about evenly between B_1 and B_2. Let us define a **split** formally:

Definition 1 *Let B_1 and B_2 be a partition of V. A set $H \subset V$ splits within d if $||H \cap B_1| - |H \cap B_2|| < d$.*

A partition (B_1, B_2) of V is called a **split partition** if the following are satisfied:

(a) $|B_1| = |B_2|$,

(b) For every v in V and for all subsets X and Y of V of size less than $20 \log \Delta$ the following sets split within $\frac{1}{4}\Delta^{11/20}$:

$$N(v), N(X), N(v) \cap N(X), \{w \in N(X) : d_{N(Y)}(w) > \Delta - 7\Delta^{39/40}\}$$

In particular, the degrees "split". Also note that condition **(a)** implies $|E(B_1)| = |E(B_2)|$. We will show in section 2.5 that for a suitably defined random split, the desired properties hold with positive probability. We will also give a straightforward linear time randomized procedure that constructs such a split. We assume from here on that a split partition (B_1, B_2) of V is given.

The goal of our edge coloring algorithm is to construct disjoint perfect matchings $M_1, ..., M_k$ such that $H = G - \cup_{i=1}^{k} M_i$ is a bipartite reduction of G contained in (B_1, B_2) (we define k below). Given such matchings, one can color the edges of G with Δ colors as follows: we use $\Delta - k$ colors to color the edges of the bipartite graph H (using the algorithm derived from König's theorem) and then we assign the remaining k colors to the disjoint matchings $M_1, ..., M_k$. We will attempt to construct these matchings in two coloring passes. If we fail to construct the desired matchings, we will show the existence of, and construct, sets X and Y such that *either* $X \subset B_1$, $Y \subset B_2$ and $|B_1| - |X| > \frac{1}{2}\Delta - \Delta^{19/20}$, *or* $X \subset B_2$, $Y \subset B_1$, $|B_2| - |X| > \frac{1}{2}\Delta - \Delta^{19/20}$, *and*, in either case, $|Y| < |X| + \Delta^{19/20}$ and

$$\forall v \in X : d_Y(v) > \frac{1}{2}\Delta - \Delta^{19/20}$$

We call such a pair (X, Y) a **fail pair** in the split partition (B_1, B_2). In section 2.5 we will present a procedure that, given a fail pair (X, Y), constructs a subgraph H of G that satisfies condition **(i)** of the regular theorem.

2.3.1 The first coloring pass

In the first coloring pass, we attempt to construct $\Delta_1 = \lceil \frac{1}{2}\Delta + \Delta^{3/4}\log(\Delta) \rceil = \frac{1}{2}\Delta + \delta$ disjoint perfect matchings $M_1, ..., M_{\Delta_1}$ such that if $F = G - M_1 - ... - M_{\Delta_1}$ then the **reject** subgraphs $R_1 = F \cap B_1$ and $R_2 = F \cap B_2$ have maximum degree at most $\Delta^{9/10}$ and $|E(R_1)| = |E(R_2)| < \frac{1}{8}\Delta^{19/10}$. We split the construction of the matchings into the initial coloring and the patching step:

The initial coloring

We start by constructing an initial coloring of $E(B_1) \cup E(B_2)$, or, in other words, initial matchings $M'_1, ..., M'_{\Delta_1}$. In order to do this we first describe a property that we require from these initial disjoint matchings.

We will say that disjoint matchings $M_1, ..., M_k$ covering the edges of some subgraph H of G (i.e. $\cup_{i=1}^{k} M_i = E(H)$) are **balanced in H** if for any

$1 \le i < j \le k$:

$$0 \le |M_j \cap E(H)| - |M_i \cap E(H)| \le 1 \tag{2.3}$$

Let M and M' be disjoint matchings in a graph H such that $|M| > |M'|$. We will use the following procedure to modify M and M' so $|M|$ is decreased by one and $|M'|$ is increased by one, without modifying $M \cup M'$, as follows:

Balancing step: We consider the connected components of $M \cup M'$ consisting of cycles and paths whose edges alternate between edges in M and edges in M'. We observe that in all alternating cycles and in all even length alternating paths, half the edges belong to M while the other half belongs to M'. In an odd length alternating path P, however, $||M \cap P| - |M' \cap P|| = 1$. We pick a path connected component P with one more edge in M than in M'; such a path must exist since $|M| > |M'|$ (see figure 2.1). We switch the color of each edge in P.

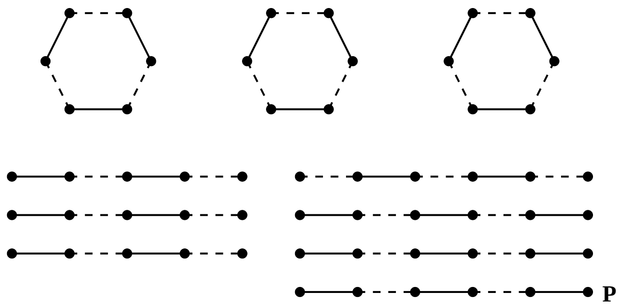

Figure 2.1: Connected components (cycles, even length and odd length paths) of subgraph defined by matchings M (full edges) and M' (dashed edges) with $|M| - |M'| = 2$

The balancing procedure

We modify any disjoint matchings $M_1, ..., M_k$ covering $E(H)$ so that inequalities 2.3 are satisfied for all $1 \le i < j \le k$ as follows:

1. We recursively apply the balancing procedure to matchings $M = M_i$ and $M' = M_j$ with largest and smallest number of edges, until all matchings have l or $l+1$ edges for some integer l.

Note that, in each iteration, we either decrease by one the difference between the largest and smallest matchings or we decrease by at least one the number of largest and smallest matchings. Thus, no more than $O(n^2)$ iterations are required.

2. We reorder the matchings so that all matchings of size $l+1$ follow the matchings of size l.

We are ready now to describe the initial coloring. We construct disjoint matchings $M'_1, ..., M'_{\Delta_1}$ balanced in B_1 and in B_2 such that $\cup_{i=1}^{\Delta_1} M'_i = E(B_1) \cup E(B_2)$ as follows:

1.1 We color $E(B_1)$ using Δ_1 (greater than $\Delta(B_1), \Delta(B_2)$) colors by applying Fournier's algorithm to obtain initial disjoint matchings $M_1^1, M_2^1, ..., M_{\Delta_1}^1$ covering $E(B_1)$; similarly we obtain disjoint matchings $M_1^2, M_2^2, ..., M_{\Delta_1}^2$ covering $E(B_2)$.

1.2 We modify the matchings obtained in **1.1** so $M_1^1, M_2^1, ..., M_{\Delta_1}^1$ are balanced in B_1 and $M_1^2, M_2^2, ..., M_{\Delta_1}^2$ are balanced in B_2.

1.3 We set $M'_i = M_i^1 \cup M_i^2$ for every $i = 1, ..., \Delta_1$.

Note that $|M'_i \cap E(B_1)| = |M'_i \cap E(B_2)|$ for every $i = 1, ..., \Delta_1$ since $|E(B_1)| = |E(B_2)|$. Let $n(B_1, i)$ and $n(B_2, i)$ be the number of vertices in B_1 and B_2, respectively, missed by M'_i, for $i = 1, ..., \Delta_1$. Then,

Claim 1 $n(B_1, i) = n(B_2, i) < 3\delta$ *for* $i = 1, ..., \Delta_1$.

Proof: Clearly, it is enough to show $n(B_1, i) < 3\delta$ for every $i = 1, ..., \Delta_1$. By the definition of a split partition no vertex will be missed by more than $\frac{9}{8}\delta$ matchings $M'_1, ..., M'_{\Delta_1}$. Since the matchings are balanced in B_1, the difference between the number of vertices in B_1 missed by any two matchings M'_i and M'_j is at most 2. Since $|B_1|$ is at most Δ, it follows that

$$n(B_1, i) = (\Delta_1)^{-1}(\Delta \frac{9}{8}\delta) + 2 \leq 3\delta$$

for every $i = 1, ..., \Delta_1$. □

The patching

Once we obtain an initial coloring $M'_1, ..., M'_{\Delta_1}$, we recursively construct the perfect matchings M_i by augmenting M'_i along vertex disjoint **patches** in $F = G - M_1 - ... - M_{i-1}$; a **patch** is a M'_i-augmenting path whose edges alternate between edges in the bipartition and edges in the matching M_i. We will actually insist that every patch has an endpoint in B_1 and the other in B_2. The edges of M'_i on the patch that get uncolored by the augmentation are added to reject graphs R_1 and R_2. If we are not successful in constructing a patch between two vertices missed by M'_i, we will show that there exists a fail pair (X, Y) in (B_1, B_2).

Let $x_1, ..., x_s$ and $y_1, ..., y_s$ be the vertices in B_1 and B_2, respectively, that are missed by M'_i. Note that $s = n(B_1, i) \leq 3\delta$. For $r = 1, ..., s$, we attempt to construct a patch P_r in F from x_r to y_r, vertex disjoint from $P_1, ..., P_{r-1}$, as follows:

2.1 If there is an edge in F between x_r and y_r, we set P_r to consist of (x_r, y_r) only. Otherwise, we define a vertex $v \in B$ to be **unavailable** if it belongs to a patch constructed in the previous $\lceil \Delta^{1/10} \rceil$ matchings $(M_{i-1}, ..., M_{i-\lceil \Delta^{1/10} \rceil})$ or if it belongs to one of $P_1, ..., P_{r-1}$. We call a vertex v **available** if v is not unavailable and is not matched in M'_i with an unavailable vertex.

2.2 Let Y^1 be the set of all available vertices in $B_2 \cap N^F(x_r)$ and let $Y^2 = \{v \in B_2 : (v, u) \in M'_i \text{ and } u \in Y^1\}$. Similarly, let X^1 be the set of all available vertices in $B_1 \cap N^F(y_r)$ and let $X^2 = \{v \in B_1 : (v, u) \in M'_i \text{ and } u \in X^1\}$. If there is an edge in F between a vertex v_x in X^2 and a vertex v_y in Y^2 we let the patch P_r be defined by the sequence of vertices $x_r, u_x, v_x, v_y, u_y, y_r$ where (u_x, v_x) and (u_y, v_y) are edges of M'_i.

If for every $i = 1, ..., \Delta_1$, we successfully construct and augment the disjoint patches between pairs of vertices missed by M'_i, we do obtain perfect matchings $M_1, ..., M_{\Delta_1}$. Note that no vertex will be incident to more than $\frac{\Delta_1}{\Delta^{1/10}} < \Delta^{9/10}$ rejected edges. Every patch contains the same number of edges in B_1 and in B_2 so that $|E(R_1)| = |E(R_2)| < 3\delta\Delta_1 < \frac{1}{8}\Delta^{19/10}$, for Δ large enough.

2.3 If there is no edge between X^2 and Y^2, we set $X = X^2$ and $Y = B_2 - Y^2$.

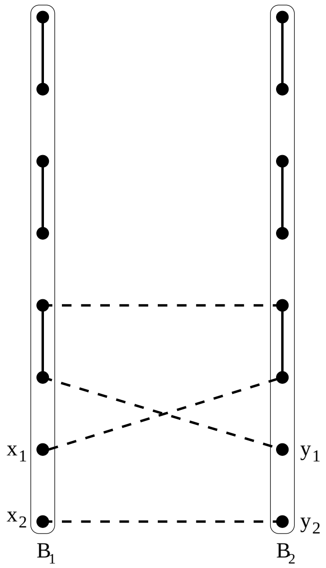

Figure 2.2: Patches of length 1 and 5

Claim 2 *(X, Y) is a fail pair in the split partition (B_1, B_2).*

Before we prove this claim, we remark that $d^F_{B_2}(v) > \frac{1}{2}\Delta - 2\Delta^{9/10}$ for every $v \in B_1$ and $d^F_{B_1}(v) > \frac{1}{2}\Delta - 2\Delta^{9/10}$ for every $v \in B_2$ since $d^F(v) = \Delta - i$ after iteration i, and $\Delta(R_1)$, $\Delta(R_2)$ are less than $\Delta^{9/10}$.

Proof: We first count the number of vertices in B_1 (B_2) that are not available. At most 3 vertices in B_1 (B_2) belong to a specific patch. Since we construct no more than 3δ patches in any one matching, the total number of vertices that are not available in iteration i is at most $2(\lceil\Delta^{1/10}\rceil 3\delta < \frac{1}{4}\Delta^{19/20}$.

To prove the claim, we must show:

(i) $|Y| < |X| + \Delta^{19/20}$

We note that $|X| = |X^1| \geq d^F_{B_1}(y_r) > \frac{1}{2}\Delta - 2\Delta^{9/10} - \frac{1}{4}\Delta^{19/20} > \frac{1}{2}\Delta - \frac{1}{2}\Delta^{19/20}$. Similarly $|Y^2| > \frac{1}{2}\Delta - \frac{1}{2}\Delta^{19/20}$. Then, $|Y| = |B_2 - Y^2| \leq \Delta - \frac{1}{2}\Delta + \frac{1}{2}\Delta^{9/10} < |X| + \Delta^{19/20}$.

(ii) $d_Y(v) > \frac{1}{2}\Delta - \Delta^{19/20}$ for every $v \in X$:

Since $d^F_{Y^2}(v) = 0$ for every $v \in X$, it follows that $d_Y(v) \geq d^F_{B_2-Y^2}(v) = d^F_{B_2}(v) > \frac{1}{2}\Delta - 2\Delta^{9/10} > \frac{1}{2}\Delta - \Delta^{9/10}$.

(iii) $|B_1| - |X| > \frac{1}{2}\Delta - \Delta^{19/20}$

Since $d^F_X(v) = 0$ for any $v \in Y^2$, it follows that $|B_1 - X| \geq d^F_{B_1-X}(v) = d^F_{B_1}(v) \geq \frac{1}{2}\Delta - 2\Delta^{9/10} > \frac{1}{2}\Delta - \Delta^{19/20}$.

□

2.3.2 The second coloring pass

After the first coloring pass, we obtain the Δ-regular reduction F of G that contains the reject graphs $R_1 = B_1 \cap F$ and $R_2 = B_2 \cap F$ of maximum degree $\Delta^{9/10}$ such that $|E(R_1)| = |E(R_2)| < \frac{1}{8}\Delta^{19/10}$. Recall that $d^F_{B_2}(v) > \frac{1}{2}\Delta - 2\Delta^{9/10}$ for every $v \in B_1$ and $d^F_{B_1}(v) > \frac{1}{2}\Delta - 2\Delta^{9/10}$ for every $v \in B_2$.

We now attempt to construct the remaining $\Delta_2 = \lceil \frac{1}{2}\Delta^{19/20} \rceil$ disjoint perfect matchings $M_{\Delta_1+1}, ..., M_{\Delta_1+\Delta_2}$ in F such that $E(R_1) \cup E(R_2) \subseteq \cup_{i=1}^{\Delta_2} M_{\Delta_1+i}$. If successful, $H = F - \cup_{i=1}^{\Delta_2} M_{\Delta_1+i}$ is clearly bipartite reduction of G. If we fail in constructing these matchings, we will show that there exists a fail pair (X, Y) in the split partition (B_1, B_2).

The initial coloring

We construct the initial matchings $M'_{\Delta_1+1}, ..., M'_{\Delta_1+\Delta_2}$, balanced in B_1 and in B_2, such that $\cup_{i=1}^{\Delta_2} M'_{\Delta_1+i} = E(R_1) \cup E(R_2)$. We construct these initial matchings as we did in the first coloring pass. We note that $|M'_{\Delta_1+i} \cap E(R_1)| = |M'_{\Delta_1+i} \cap E(R_2)| \leq \frac{1}{4}\Delta^{19/20}$ for every $i = 1, ..., \Delta_2$ since $|E(R_1)| = |E(R_2)| < \frac{1}{8}\Delta^{19/10}$.

The patching

We recursively construct M_{Δ_1+i} by augmenting M'_{Δ_1+i} in $H = F - M_{\Delta_1+1} - ... - M_{\Delta_1+(i-1)}$ as follows:

2.1 Let U_1 and U_2 be the sets of vertices missed by M'_{Δ_1+i} in B_1 and B_2, respectively. Note that $|U_1| = |U_2|$. We attempt to find a perfect matching M^* in the bipartite graph $(U_1, U_2) \cap H$. If successful, we simply add M^* to M'_{Δ_1+i} to obtain M_{Δ_1+i}.

2.2 If there is no perfect matching in $(U_1, U_2) \cap H$, there must exist $X \subseteq U_1$ such that $|X| > |N^H_{U_2}(X)|$. We set $Y = N^F_{U_2}(X) \cup F_2$, where F_2 is the set of endpoints of matching edges of M'_{Δ_1+i} in B_2.

Claim 3 *(X, Y) is a fail pair in the split partition (B_1, B_2).*

Proof: Since $|M'_{\Delta_1+i} \cap E(R_2)| \leq \frac{1}{4}\Delta^{19/20}$, it follows that $|Y| < |X| + |F_2| < |X| + \Delta^{19/20}$. In addition, $d_Y(v) = d^H_{B_2}(v) \geq d^F_{B_2}(v) - \Delta_2 > \frac{1}{2}\Delta - \Delta^{19/20}$. Finally since $|X| > |N^H_{U_2}(X)|$ there exist $v \in U_2$ such that $d^H_X(v) = 0$. Since $d^H_{B_1}(v) > \frac{1}{2}\Delta - \Delta^{19/20}$, the claim follows. $\square$

2.4 Proof of the main theorem: an overview

While far more complicated, our proof of the main theorem uses a similar approach as the proof of the regular theorem we just presented. We give the details in chapters 3, 4, 5 and 6. We describe here the difficulties we will encounter and the methods we will develop while extending the regular case proof to the main theorem; we hope the reader will thus appreciate the complexity of the task, understand the necessicity for this long and technical proof and, perhaps, become interested and motivated to read through the technical details!

The main theorem generalizes the regular theorem in two ways. First, it applies to any simple graph $G = (V, E)$ of maximum degree $\Delta \geq \Delta_0$, not just Δ-regular graphs. Second, it applies to all graphs with $\Delta \geq \frac{1}{6}|V|$, a far larger proportion than the proportion of graphs satisfying $\Delta \geq \frac{1}{2}|V|$. We discuss the second generalization first, as it poses no true difficulty.

In fact, when extending the proof of the regular theorem so it holds for $\Delta \geq \frac{1}{6}|V|$, the only substantial change is that we allow longer patches in the first coloring pass. Given two vertices $x \in B_1$ and $y \in B_2$ missed by an initial matching M'_i, we will attempt to construct a patch between x and y, i.e. an M'_i-augmenting path alternating between edges in M'_i and edges in (B_1, B_2), of length up to $4\Delta^{1/20} + 1$ (see figure 2.3). If such a patch does not exist, we will show the existence of and construct a fail pair (X, Y) in the bipartition (B_1, B_2); in a way, a fail pair is a bottleneck that prevents two vertices to be connected by a patch. We then use this fail pair to construct the subgraph H of G that satisfies **(i)** of the regular theorem.

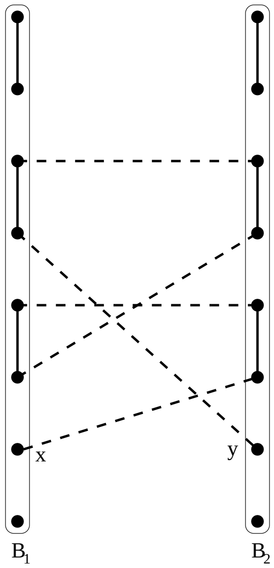

Figure 2.3: A patch of length 9 connecting vertices x and y missed by an initial matching n $E(B_1) \cup E(B_2)$

It may at first seem surprising that in extending the regular theorem to graphs with $\Delta \geq \frac{1}{6}|V|$, we did not need to add a condition forbidding G to contain an overfull subgraph of maximum degree Δ. After all, if $\Delta < \frac{1}{2}|V|$ then G could contain an overfull subgraph H of maximum degree Δ and thus G would not be Δ edge colorable. In this case, however, the overfull subgraph H must contain a subgraph satisfying **(i)** of the regular theorem. To see this, note first that H must contain a subgraph H' of minimum degree $\Delta - 2\sqrt{\Delta}$. Furthermore, since G is Δ-regular, $V - H$ must also be an overfull subgraph of maximum degree Δ, implying $|V - H'| \geq |V - H| > \Delta$. H' would thus satisfy condition **(i)** of the regular theorem. Thus, in the regular case we do not need to worry about overfull subgraphs because none exist unless condition **(i)** holds.

In a similar vein, when extending the regular theorem to non-regular graphs, we take advantage of the fact that if G has an overfull subgraph F, but has no subgraph H satisfying **(i)** of the main theorem, then F has a very special structure. To understand this, we divide the vertices of G into **big** and **small** vertices, i.e. into $B = \{v \in V : d(v) > \frac{1}{2}\Delta\}$ and $S = \{v \in V : d(v) \leq \frac{1}{2}\Delta\}$. Clearly, F has at most one small vertex. In fact, we can impose conditions on the graph G so that F contains all but one big vertex. To do so, we note that by Vizing's adjacency lemma, it is enough to prove the main theorem for graphs G such that the following property holds for every $u, v \in V$:

$$d(u) + |\{x \in V : (x,v) \in E \text{ and } d(x) = \Delta\}| \geq \Delta + 1$$

We will call a graph G **Vizing** if this property is satisfied for all $u, v \in V$. We observe that, if G is Vizing, then **(A)** S is a stable set and **(B)** $d_B(v) > \frac{1}{2}\Delta$ for every $v \in B$. Now suppose F is an overfull subgraph of G of maximum degree Δ and that $B - F$ contains between 3 and $\frac{1}{2}\Delta - 3$ elements. By **(B)**, each vertex in $B - F$ sees $\frac{1}{2}\Delta - |B - F|$ vertices in $B \cap F$. Hence there exists $(\frac{1}{2}\Delta - |S|)|S| \geq \Delta$ edges between F and $B - F$, contradicting the fact that F is overfull. Similar arguments show that $|B - F| \neq 2$, and also that $|B - F| \not\geq \frac{1}{2}\Delta - 3$ unless condition **(i)** holds.

Now, just as the big vertices play the major role in any possible overfull subgraph, they also present the only real difficulty in proving the theorem. Thus, for the moment, we assume there are no small vertices. In our constructive proof of the main theorem, we will attempt to obtain disjoint matchings $M_1, ..., M_k$ such that $H = G - \cup_{i=1}^{\Delta_1+\Delta_2} M_i$ is a reduction of G that is easily $\Delta(H)$ edge colorable (H will usually be a bipartite graph). The procedure is quite similar to that used in the regular case: we obtain each M_i by augmenting along patches constructed bewteen some of the vertices missed by an initial matching M_i'. A major difference from the regular case is that the matchings $M_1, ..., M_k$ will NOT necessarily be perfect. For one thing, $|V|$ may be odd. More to the point, our patching technique for augmenting initial matchings in both coloring passes relies on keeping the degree of each vertex across the bipartition high throughout the algorithm. So, we must develop a methodology for choosing which vertices are missed by which matching. We note that the number of times a vertex can be missed by a matching depends on the difference between its degree and Δ. Thus, it is not surprising that in making our choices, we consider the following notion:

Definition 2 *The* **deficiency** *of a vertex $v \in B$ is def$(v) = \Delta - d_B(v)$.*

We also find it useful to extend the notion of deficiency to subsets of vertices: if $H \subset B$ then $\text{def}(H) = \sum_{v \in H} \text{def}(v)$.

If the deficiency of G is large, which we will define as having more than $2\Delta^{9/10}$ vertices of deficiency greater than $\Delta^{9/10}$, it is not too hard to choose the vertices to be missed in both coloring passes: all vertices that are missed by the initial matchings except the "large degree" big vertices. The patch construction itself, however, is more complex than in the regular case because of the possible large number of low degree big vertices; we omit the details in this sketch.

If no more than $2\Delta^{9/10}$ vertices have deficiency greater than $\Delta^{9/10}$, i.e. the set of vertices $B^- = \{v \in B : \mathrm{def}(v) > \Delta^{9/10}\}$ is no larger than $2\Delta^{9/10}$, then the minimum degree of $B - B^-$ is $\Delta - 2\Delta^{9/10}$. As $B - B^-$ is "almost" regular and "almost" equal to the whole graph (recall that we assume there are no small vertices), we will construct all patches through the vertices of $B - B^-$ only, using patching techniques similar to the ones we used in the regular case. So, patching is "essentially" done. The problem now is to choose what vertices, missed by an initial matching, to patch. The constraints dictating our choices are:

(i) No vertex should be missed by more than $\mathrm{def}(v)$ matchings (to obtain a reduction), and

(ii) No vertex should belong to too many patches, so that the degree accross the bipartition remains high (necessary for patching) and so that the degree of the rejects graph remains low (necessary for the reject coloring pass),

(iii) The total number of rejected edges should be split evenly (or just about) between B_1 and B_2 (to facilitate the reject coloring pass).

We can satisfy the first two constraints by initialy choosing, for each $v \in B^*$, about $\frac{1}{2}\mathrm{def}(v)$ initial matchings that miss v and deciding that the corresponding M_i miss v. To satisfy the third constraint, we will develop several methods to modify our initial "choices" so that, in each initial matching, the number of vertices we will need to patch is split evenly between B_1 and B_2.

If the input graph G has **medium deficiency**, by which we mean that $\mathrm{def}(B) \geq \Delta^{12/10}$ but no more than $2\Delta^{9/10}$ vertices have deficiency greater than $\Delta^{9/10}$, we can easily modify the initial choices, essentially because there are many of them.

The smaller the deficiency, the harder it is to insure the third constraint. This is especially true if small vertices are present. For this reason, if

$\mathrm{def}(B) < \Delta^{12/10}$, we find it useful to identify small vertices, while keeping multiple edges, and "create" new big vertices of degree greater than $\frac{1}{2}\Delta$ but at most Δ, of course. This identification process leaves at most 3 small vertices and thus very little fake deficiency, but it "creates" new real deficiency at the new big vertices. We now resume our assumption that no small vertices are present.

In G has **small deficiency**, i.e. when $2\Delta < \mathrm{def}(B) < \Delta^{12/10}$, very few vertices will be missed by each matching. We will develop more sophisticated techniques to finalize our choices. These techniques fail, however, on the lowest deficiency graphs, as they rely on a certain number of vertices being missed by each matching.

In the **smallest** deficiency case, when $\mathrm{def}(G) \leq 2\Delta$, we must very carefully choose what vertices are going to be missed by a specific matching. However, even with special care, we will not always be able to construct a bipartite reduction. We illustrate this with the following example. Suppose $|B|$ is even, $|S| = 0$, $\mathrm{def}(B) = \Delta$, $\mathrm{def}(b_1) = \frac{1}{2}\Delta - 2$, $\mathrm{def}(b_2) = \mathrm{def}(b_3) = \frac{1}{4}\Delta + 1$ where $b_1, b_3 \in B_2$ and $b_2 \in B_1$. Then $c_B = |E(B_1)| - |E(B_2)| = \frac{1}{2}(\mathrm{def}(B_2) - \mathrm{def}(B_1)) = \frac{1}{4}\Delta - 1$. If $M_1, ..., M_k$, for $k = \frac{1}{2}\Delta + o(\Delta)$, were matchings whose removal leaves a bipartite reduction $H = G - M_1 - ... - M_{\Delta_1+\Delta_2}$ in (B_1, B_2), then both b and b'' must be missed simultaneously by exactly c_B of these matchings. On the other hand, all $d_{B_2}(b'') = \frac{3}{8}\Delta + o(\Delta)$ edges incident to b'' must also be covered by the union of the matchings. We would thus require $k \geq c_B + d_{B_2}(b'') > \frac{5}{8}\Delta + o(\Delta)$, a contradiction. We thus cannot obtain a bipartite reduction with so few matchings. Instead, we must satisfy ourselves with H being a near-bipartite reduction N with no overfull subgraph of maximum degree $\Delta(N)$. We recall that a polynomial time algorithm by Reed [Ree95] gives us a tool to color such a near- bipartite graph N with $\Delta(N)$ colors. For completenes, we include this algorithm in the appendix (A).

2.5 The split partition and the forbidden subgraph

We present the two technical procedures omitted from the proof of the regular theorem: the construction of a split partition and the construction of a forbidden subgraph from a fail pair. We choose to describe these procedures in the general setting of a non-regular Vizing graph $G = (V, E) = (B \cup S, E)$ of maximum degree $\Delta \geq \frac{1}{6}|B \cup S|$.

2.5.1 The split partition

Our edge-coloring algorithms require that a special vertex partition $(B_1 \cup S_1, B_2 \cup S_2)$ of $B \cup S$ be provided along with the input graph $G = (B \cup S, E)$. In particular, we insist that the degree of each vertex is split about evenly between the two sides of the bipartition and that B and S are split about evenly as well. Furthermore, just in case our algorithms fail, we need additional sets of vertices to split about evenly between $B_1 \cup S_1$ and $B_2 \cup S_2$: this enables us to construct, in the forbidden subgraph construction procedure, a forbidden subgraph of G if we find two sets X and Y such that either

$$X \subset B_1 \text{ and } |B_1| - |X| > \frac{1}{4}\Delta - \Delta^{19/20} \tag{2.4}$$

or

$$X \subset B_2 \text{ and } |B_2| - |X| > \frac{1}{4}\Delta - \Delta^{19/20} \tag{2.5}$$

and, in both cases,

$$\begin{aligned} |Y| &< |X| + \Delta^{19/20} && (2.6)\\ d_Y(v) &> \frac{1}{2}\Delta - \Delta^{19/20} \text{ for all } v \in X && (2.7) \end{aligned}$$

We call (X, Y) a **fail pair** in $(B_1 \cup S_1, B_2 \cup S_2)$. Our edge-coloring algorithms fail to Δ edge-color G only if a fail pair (X, Y) is found. We now recall the definition of **splitting** and of a split partition.

Definition 3 *Let $(B_1 \cup S_1, B_2 \cup S_2)$ be a partition of $B \cup S$. A set $H \subset B \cup S$ splits within d if $||H \cap (B_1 \cup S_1)| - |H \cap (B_2 \cup S_2)|| < d$.*

Let $b_1, b_2, b_3, b_4, ..., b_k$ be the vertices in B such that $\mathrm{def}(b_1) \geq \mathrm{def}(b_2) \geq \mathrm{def}(b_3) \geq \mathrm{def}(b_4) \geq ... \geq \mathrm{def}(b_k)$ and let $s_1, s_2, ..., s_k$ be the vertices in S such that $d(s_1) \geq d(s_2) \geq ... \geq d(s_l)$. A partition $(B_1 \cup S_1, B_2 \cup S_2)$ of $B \cup S$ is called a **split partition** if the following properties are satisfied:

(a) $B = B_1 \cup B_2$, $S = S_1 \cup S_2$, $0 \leq |B_1| - |B_2| \leq 1$ and $0 \leq |S_2| - |S_1| \leq 1$.

(b) For all v in $B \cup S$ and for all $X, Y \subset B$ of size less than $20 \log \Delta$ the following sets split within $\frac{1}{4}\Delta^{11/20}$: $N_B(v), N_S(v), N_B(X), N_B(v) \cap N_B(X), \{w \in N_B(X) : d_{N_B(Y)}(w) > \Delta - 7\Delta^{39/40}\}$

(c) If $|B|$ is odd, either $b_{2i} \in B_1$ and $s_{2i+1} \in B_2$ or $b_{2i} \in B_2$ and $b_{2i+1} \in B_1$ for $i = 1, ..., \lfloor\frac{|B|}{2}\rfloor$; in addition, $b_1, b_3 \in B_1$, $b_2 \in B_2$ and $\mathrm{def}(B_1 - b_1 - b_3) \leq \mathrm{def}(B_2 - b_2)$. If $|B|$ is even, either $b_{2i-1} \in B_1$ and $b_{2i} \in B_2$ or $b_{2i-1} \in B_2$ and $b_{2i} \in B_1$ for $i = 1, ..., \frac{|B|}{2}$; in addition, $b_1, b_3 \in B_2$, $b_2, b_4 \in B_1$ and $\mathrm{def}(B_1 - b_2 - b_4) \leq \mathrm{def}(B_2 - b_1 - b_3)$.

(d) If $|S|$ is odd, either $s_{2i} \in S_1$ and $s_{2i+1} \in S_2$ or $s_{2i} \in S_2$ and $s_{2i+1} \in S_1$ for $i = 1, ..., \lfloor\frac{|S|}{2}\rfloor$; in addition, $s_1 \in S_2$. If $|S|$ is even and non-empty, either $s_{2i-1} \in S_1$ and $s_{2i} \in S_2$ or $s_{2i-1} \in S_2$ and $s_{2i} \in S_1$ for $i = 1, ..., \frac{|S|}{2}$; in addition, $s_1 \in S_2$ and $s_2 \in S_1$.

Let $c_B = |E(B_1)| - |E(B_2)|$. Note that property **(a)** implies that if $|B|$ is odd then $c_B = \frac{1}{2}(\Delta - (\mathrm{def}(B_1) - \mathrm{def}(B_2)))$, and if $|B|$ is even then $c_B = \frac{1}{2}(\mathrm{def}(B_2) - \mathrm{def}(B_1))$. By property **(c)**, it follows that $0 \leq \mathrm{def}(B_1) - \mathrm{def}(B_2) \leq \mathrm{def}(b_1)$ and $\frac{1}{4}\Delta < \frac{1}{2}(\Delta - \mathrm{def}(b_1)) \leq c_B \leq \frac{1}{2}\Delta$ if $|B|$ is odd and $0 \leq \mathrm{def}(B_2) - \mathrm{def}(B_1) \leq \mathrm{def}(b_1)$ and $0 \leq c_B \leq \frac{1}{2}\mathrm{def}(b_1) < \frac{1}{4}\Delta$ if $|B|$ is even. We also observe that $0 \leq c_S = \frac{1}{2}(d(S_2) - d(S_1)) \leq \frac{1}{2}d(s_1) \leq \frac{1}{4}\Delta$. Finally, we note that $\Delta(B_1 \cup S_1)$ and of $\Delta(B_2 \cup S_2)$ are both less than $\frac{1}{2}\Delta + \frac{1}{4}\delta$.

The following procedure constructs with positive probability a split partition $(B_1 \cup S_1, B_2 \cup S_2)$ of the set of vertices $B \cup S$ of G:

Partition Step: We order the vertices in B by non-decreasing deficiency (i.e non-increasing degree within B). For each successive ordered pair of vertices we switch the order of the pair with probability $\frac{1}{2}$ and put the first vertex in the set B_1 and the second in the set B_2. If $|B|$ is even, after all the vertices but b_1, b_2, b_3 and b_4 have been assigned to B_1 or B_2, we rename B_1 and B_2 so that $\mathrm{def}(B_1) \leq \mathrm{def}(B_2)$ and we add b_1 and b_3 to B_2 and b_2 and b_4 to B_1. If $|B|$ is odd, after all the vertices but b_1, b_2 and b_3 have been assigned to B_1 or B_2, we rename B_1 and B_2 so that $\mathrm{def}(B_1) \leq \mathrm{def}(B_2)$ and we add b_1, b_3 to B_1 and b_2 to B_2.

We similarly split S into sets S_1 and S_2. The ordering of the vertices is by non-decreasing degree. If $|S|$ is even, after all the vertices but the last pair have been assigned to S_1 or S_2, we rename S_1 and S_2 so that $d(S_1) \leq d(S_2)$ and we add s_1 to S_2 and s_2 to S_1. If $|S|$ is odd, after all the vertices but s_1 have been assigned to S_1 or S_2, we rename S_1 and S_2 so that $d(S_1) \leq d(S_2)$ and we add s_1 to S_2.

It is easy to see that the resulting partition $(B_1 \cup S_1, B_2 \cup S_2)$ of $B \cup S$ satisfies properties **(a)**, **(c)** and **(d)** of a split partition. In addition, property **(b)** is satisfied with probability at least $\frac{1}{2}$, as we show below. We note that the running time of this procedure is linear in the size of the vertex set, and that we can obtain a split partition deterministically in $O(2^n)$ time by exhaustively testing every possible partition of $B \cup S$.

Claim 4 *The probability that there is some v in $B \cup S$ and some subsets X and Y of B such that $|X|, |Y| < 20 \log n$ for which one of the following sets fails to split is less than $\frac{1}{2}$:*

$$N_B(v), N_S(v), N(X), N(v) \cap N(X), \{w \in N(X) : d_{N(Y)}(w) > \Delta - 7\Delta^{39/40}\}$$

Proof: Let $|B \cup S| = n$ and let $m = 20 \log n$. There are fewer than

$$2n + m\binom{n}{m} + nm\binom{n}{m} + m^2\binom{n}{m}\binom{n}{m} < n^{41 \log n}$$

sets that we want to split. Let $H = \{v_1, ..., v_k\}$ be one of them. We assume that no two vertices in H are paired in the partition step (if such pairs exist, they split evenly and we only need to worry about the remaining vertices). Let $H_1 = H \cap (B_1 \cup S_1)$ and $H_2 = H \cap (B_1 \cup S_2)$. We define for all $v_i \in H - \{b_1, b_2, b_3, b_4, s_1, s_2\}$ the random variable X_i:

$$X_i = \begin{cases} -1 & \text{if } v_i \in H_2 \\ 1 & \text{if } v_i \in H_1 \end{cases}$$

Then $||H_2| - |H_1|| \leq |\sum_{i=1}^{k} X_i| + 3$, and

$$\Pr(|\sum_{i=1}^{k} X_i| > \frac{1}{4}\sqrt{k}\Delta^{1/20} - 3) < 2e^{\frac{-k\Delta^{1/10}}{32k}} < (2n^{41 \log n})^{-1}$$

since $\Delta \geq \frac{n}{6}$ and Δ is large enough. $\square$

2.5.2 The forbidden subgraph construction

If we find a fail pair (X, Y) in the split partition $(B_1 \cup S_1, B_2 \cup S_2)$, then we can construct from it a subgraph satisfying condition **(i)** of the main (and regular) theorem. To show this, we first simplify our notation by calling a subgraph of B **forbidden** if the minimum degree of H is at least $\Delta - \Delta^{79/80}$ and either:

(i) H is bipartite, or

(ii) $|B - H| > \frac{1}{2}\Delta - \Delta^{79/80}$

If H satisfies **(ii.)** we call it **type 2**, otherwise we call it type **type 1**. Note that if H is type **1** then $|B - H| \leq \frac{1}{2}\Delta - \Delta^{79/80}$.

We observe that, in the case of a regular graph $G = (B, E)$ with $|B| < 2\Delta$, if B contains a forbidden subgraph H then condition **(i)** of the regular theorem follows, for if H is of type 2 then $|B - H| > \Delta - 2\Delta^{79/80}$. A slightly less trivial argument, which we omit until the proof of the main theorem, shows that condition **(i)** of the main theorem also follows if a general Vizing graph $G = (B \cup S, E)$ with $|B \cup S|$ contains a forbidden subgraph.

Lemma 13 (The patching lemma) *Let $G = (B \cup S, E)$ be a Vizing graph of maximum degree $\Delta \geq \frac{1}{6}|B \cup S|$ and let $(B_1 \cup S_1, B_2 \cup S_2)$ be a split partition of $B \cup S$. If (X, Y) is a fail pair in $(B_1 \cup S_1, B_2 \cup S_2)$ then B contains a forbidden subgraph.*

We prove the patching lemma by showing, in claims 5, 6 and 7, that the forbidden subgraph construction procedure, described below, returns a forbidden subgraph G if G contains a fail pair (X, Y). By symmetry, we can and will assume that $X \subset B_1$ and $|B_1| - |X| > \frac{1}{4}\Delta - \Delta^{19/20}$, and we recall that the following hold:

$$|Y| < |X| + \Delta^{19/20} \tag{2.8}$$

$$d_Y(v) > \frac{1}{2}\Delta - \Delta^{19/20} \text{ for all } v \in X \tag{2.9}$$

The forbidden subgraph construction procedure

1. We construct the set $Z = \{v \in Y : d_X(v) \geq \frac{1}{2}\Delta - 3\Delta^{39/40}\}$.

Since $|Y - Z| < \frac{5}{2}\Delta^{39/40}$ (by claim 5 below), it follows that $d_Z(v) \geq \frac{1}{2}\Delta - 3\Delta^{39/40}$ for all $v \in X$ and that $d_X(v) \geq \frac{1}{2} - 3\Delta^{39/40}$ for all $v \in Z$. Note that $X \subset B_1$ and $Z \subset B_2$ and that

$$|B_2 - Z| \geq |B_2 - Y| \geq \frac{1}{4}\Delta - 3\Delta^{19/20} \tag{2.10}$$

In the last two steps of this procedure, we will use X and Z to construct a forbidden subgraph in B. As the construction is entirely within the subgraph B, and to simplify the notation, we will use $N(v)$ to denote $N_B(v)$ in these last two steps and the remainder of this section.

2. We construct a set $X^0 \subset X$ such that $Z \subset N(X^0)$ by recursively picking $x \in X - X^0$ so that $|N_Z(x) - N_Z(X^0)|$ is largest. We similarly construct $Z^0 \subset Z$ such that $X \subset N(Z^0)$.

We show in claim 6 that $|X^0|$ and $|Z^0|$ are less than $20 \log \Delta$. Since every $x \in X^0$ (respectively, $x \in Z^0$) is adjacent to at most $3\Delta^{39/40} + \Delta^{11/20}$ vertices in $B_2 - Z$ (respectively, $B_1 - X$), it follows that

$$|(N(X^0) \cap B_2) - Z| < 80\Delta^{39/40} \log \Delta \tag{2.11}$$
$$|(N(Z^0) \cap B_1) - X| < 80\Delta^{39/40} \log \Delta \tag{2.12}$$

3. We construct sets $K_X = \{v \in N(X^0) : d_{N(Z^0)}(v) > \Delta - 7\Delta^{39/40}\}$ and $K_Z = \{v \in N(Z^0) : d_{N(X^0)}(v) > \Delta - 7\Delta^{39/40}\}$.

In claim 7, we prove that if $K_X \cap K_Z = \emptyset$ then $K_X \cup K_Z$ is a forbidden subgraph of G of type 1, and if $K_X \cap K_Z \neq \emptyset$, $K_X \cap K_Z$ is a forbidden subgraph of G of type 2.

The patching lemma then follows.

Claim 5 *After step 1,* $|Y - Z| < \frac{5}{2}\Delta^{39/40}$.

Proof: The number of edges between vertices in X and Y is

$$\begin{aligned}
|E(X,Y)| &\geq |X|(\frac{\Delta}{2} - \Delta^{19/20}) \\
&> (|Y| - \Delta^{19/20})(\frac{1}{2}\Delta - \Delta^{19/20}) \\
&> |Y|(\frac{1}{2}\Delta - \Delta^{19/20}) - \Delta^{19/20}|Y| \\
&> |Y|(\frac{1}{2}\Delta - 2\Delta^{19/20})
\end{aligned}$$

If $|Y - Z|$ were greater than $\frac{5}{2}\Delta^{39/40}$, we would obtain the following contradiction:

$$\begin{aligned}
|E(X,Y)| &\leq |Z|(\frac{1}{2}\Delta + \Delta^{11/20}) + |Y - Z|(\frac{1}{2}\Delta - 3\Delta^{39/40}) \\
&< |Y|\frac{1}{2}\Delta - 7\Delta^{78/40} \\
&< |Y|(\frac{1}{2}\Delta - 2\Delta^{19/20})
\end{aligned}$$

since $|Y| \leq |B_2 \cup S_2| \leq \frac{1}{2}|B \cup S| \leq 3\Delta$.

□

Claim 6 *After step 2,* $|X^0| \leq 20 \log \Delta$.

Proof: Each vertex in Z is adjacent to at least $\frac{1}{2}\Delta - 3\Delta^{39/40}$ vertices in X and $|X| < 3\Delta$. It follows that each vertex in $Z - N_Z(X^0)$ is adjacent to at least $\frac{1}{2}\Delta - 3\Delta^{39/40} > \frac{1}{3}\Delta$ vertices in $X - X^0$ and $|X - X^0| < 3\Delta$. Thus we know that $Z - N_Z(X^0)$ is reduced by at least a ninth at each iteration. So there will be at most $\log_{\frac{9}{8}} \Delta < 20 \log \Delta$ iterations. □

Claim 7 *After step 3, if* $K_X \cap K_Z = \emptyset$ *then* $K_X \cup K_Z$ *is a forbidden subgraph of* G *of type 1, and if* $K_X \cap K_Z \neq \emptyset$, $K_X \cap K_Z$ *is a forbidden subgraph of* G *of type 2.*

Proof: We first show that $X \subset K_Z \cap B_1$ and, by symmetry, $Z \subset K_X \cap B_2$. If $v \in X$ then $v \in N(Z^0)$ and $d_Z(v) > \frac{1}{2}\Delta - 3\Delta^{39/40}$. Since Z is a subset of $N(X^0) \cap B_2$, it follows that $|N(v) \cap N(X^0) \cap B_2| > \frac{1}{2}\Delta - 3\Delta^{39/40}$ and, because $N(v) \cap N(X^0)$ splits (by claim 4), $|N(v) \cap N(X^0)| > \Delta - 7\Delta^{39/40}$. Thus $v \in K_Z \cap B_1$.

By inequalities 2.11 and refeq:construct3, it follows that $|N(X^0) - K_X) \cap B_2| < 80\Delta^{39/40} \log \Delta$ and $|N(Z^0) - K_Z) \cap B_1| < 80\Delta^{39/40} \log \Delta$. Because $N(X^0)$, $N(Z^0)$, K_X and K_Z split (by claim 4) and since $K_X \subset N(X^0)$ and $K_Z \subset N(Z^0)$ we obtain

$$|(N(X^0) - K_X) \cap B_1| < 81\Delta^{39/40} \log \Delta$$
$$|(N(Z^0) - K_Z) \cap B_2| < 81\Delta^{39/40} \log \Delta$$

which implies

$$|N(X^0) - K_X| < 162\Delta^{39/40} \log \Delta$$
$$|N(Z^0) - K_Z| < 162\Delta^{39/40} \log \Delta$$

so $|N(X^0) \cap N(Z^0) - K_X \cap K_Z| < 324\Delta^{39/40} \log \Delta$.

If $K_X \cap K_Z \neq \emptyset$ we obtain, from the above analysis and using inequalities 2.10, 2.11 and 2.12,

$$\begin{aligned} |K_X \cap K_Z| &< |N(X^0) \cap N(Z^0)| \\ &\leq |N(Z^0) \cap B_1| + |N(X^0) \cap B_2| \\ &\leq |X| + |Z| + 160\Delta^{39/40} \\ &< |B| - \frac{1}{2}\Delta + \Delta^{79/80} \end{aligned}$$

Furthermore, for every $v \in K_x \cap K_z$,

$$\begin{aligned} d_{K_X \cap K_Z}(v) &> d_{N(X^0) \cap N(Z^0)}(v) - 324\Delta^{39/40} \log \Delta \\ &> d_{N(X^0)}(v) - 7\Delta^{39/40} - 324\Delta^{39/40} \log \Delta \\ &> \Delta - \Delta^{79/80} \end{aligned}$$

So, $K_X \cap K_Z$ is a forbidden subgraph of G of type 2.

If $K_X \cap K_Z = \emptyset$, we obtain that $d_{K_Z}(v) > \Delta - 7\Delta^{39/40} - 162\Delta^{39/40} \log \Delta > \Delta - \Delta^{79/80}$ for every $v \in K_X$ and, by symmetry, $d_{K_X}(v) > \Delta - \Delta^{79/80}$ for every $v \in K_Z$. Thus, (K_X, K_Z) is a forbidden subgraph of G of type 1. $\square$

Chapter 3

The main theorem

3.1 Restating the theorem

We now prove the main theorem:

Theorem 14 (main theorem) *There exists Δ_0 such that for all simple graphs $G = (V, E)$ with maximum degree $\Delta \geq \Delta_0$ and $n = |V| \leq 6\Delta$, one of the following is true:*

(i) *G contains a subgraph H such that $\delta(H) \geq \Delta - \Delta^{79/80}$ and either:*

H is bipartite, or
$|V - H| > \Delta - 8\Delta^{159/160}$

(ii) *G contains an overfull subgraph H of maximum degree Δ,*

(iii) *G is Δ edge colorable.*

Furthermore, there is a procedure which runs in $O(2^n)$ time that will output either a Δ edge coloring of G or a subgraph H of G that satisfies one of (i) or (ii).

The lower bound on the maximum vertex degree Δ_0 satisfies a number of inequalities that appear throughout this paper. In order to prove the main theorem, we make its statement more precise. We need the following result by Vizing [Viz64]:

Lemma 15 (Vizing's Adjacency Lemma) *Let $G = (V, E)$ be a simple graph of maximum degree Δ such that $G - (u, v)$ is Δ edge colorable, for some edge $(u, v) \in E$. If*

$$d(u) + |\{x \in V : (x, v) \in E \text{ and } d(x) = \Delta\}| < \Delta + 1$$

then G is also Δ edge colorable. Furthermore, a Δ edge coloring of $G - (u, v)$ is extendable to a Δ edge coloring of G in $O(n^2)$ time.

This lemma motivates the following definition:

Definition 4 *A simple graph $G = (V, E)$ of maximum degree Δ is called* **Vizing** *if for all (u, v) in E:*

$d(u) + |\{x \in V : (x, v) \in E \text{ and } d(x) = \Delta\}| \geq \Delta + 1$

Given an arbitrary simple graph G we define the **Vizing reduction** of G to be the subgraph obtained by recursively removing edges (u, v) such that

$$d(u) + |\{x : (x, v) \in E \text{ and } d(x) = \Delta\}| < \Delta + 1$$

Obviously, the Vizing reduction of G is unique and Vizing. The adjacency lemma shows that we can extend a Δ-edge coloring of the Vizing reduction of G to a Δ-edge coloring of G in $O(n^4)$ time. Furthermore if the Vizing reduction of G contains a subgraph H that is overfull with maximum degree Δ or satisfies one of **(i)** or **(ii)** of the main theorem, G does too. Thus it suffices to prove the theorem for Vizing graphs.

Note that in a Vizing graph, the set $\boldsymbol{S = \{v : d(v) \leq \frac{1}{2}\Delta\}}$ is a stable set. Furthermore, if a vertex in $\boldsymbol{B = V - S}$ is adjacent to a vertex in S then it has more than $\frac{1}{2}\Delta$ neighbors of maximum degree. Thus, every vertex in B has more than $\frac{1}{2}\Delta$ neighbors in B. We shall often speak of the Vizing graph $\boldsymbol{G = (B \cup S, E)}$ rather than $G = (V, E)$ and call the vertices in B **big** and vertices in S **small**. We also call edges in $E(B)$ big and edges in $E(B, S)$ small. Finally we will often abuse notation and denote by B the graph induced by the set of big vertices of G; in general, we will denote by H the subgraph induced by vertices in the subset $H \subset B \cup S$ or we denote by H the vertices of a subgraph of G.

We now give the first refinement of the theorem. To do this we need the following definition:

Definition 5 *Let $G = (B \cup S, E)$ be a Vizing graph. We call a subgraph H of B* **forbidden** *if its minimum degree $\delta(H) \geq \Delta - \Delta^{79/80}$ and either:*

(i) *H is bipartite, or*

(ii) $|B - H| > \frac{1}{2}\Delta - \Delta^{79/80}$

If H satisfies **(ii.)** we call it of **type 2**, otherwise we call it of **type 1**. Note that if H is type **1** then $|B - H| \leq \frac{1}{2}\Delta - \Delta^{79/80}$.

Theorem 16 (first refinement) *There exists Δ_0 such that for all Vizing graphs $G = (B \cup S, E)$ with maximum degree $\Delta \geq \Delta_0$ and $|B \cup S| \leq 6\Delta$, one of the following is true:*

(i) *B contains a forbidden subgraph.*

(ii) *G contains an overfull subgraph of maximum degree Δ,*

(iii) *G is Δ edge colorable.*

Furthermore, there exists a procedure which runs in $O(2^{|V|})$ time that will output either a Δ edge coloring of G or a subgraph H of G that is forbidden or overfull with maximum degree Δ.

Claim 8 *Let $G = (B \cup S, E)$ be a Vizing graph of maximum degree Δ such that $|B \cup S| \leq 6\Delta$. If B contains a forbidden subgraph H of type 2, then $|B \cup S - H| > \Delta - 8\Delta^{159/160}$.*

Proof: We consider the set $C = \{v \in B - H : d_H(v) > \sqrt{3}\Delta^{159/160}\}$. We observe that since $|H| < 6\Delta$, then $|E(H, B \cup S - H)| < 6\Delta\Delta^{79/80} < 6\Delta^{159/80}$, and it follows that $|C| < 2\sqrt{3}\Delta^{159/160}$. If there is a vertex v of degree Δ in $B - H - C$ then $d_{B \cup S - H}(v) > \Delta - \sqrt{3}\Delta^{159/160}$ and so $|B \cup S - H| > \Delta - \sqrt{3}\Delta^{159/160}$. If no vertex of $B - H - C$ has degree Δ, then, for any $v \in B - H - C$, $|\{x \in B \cup S : (x, v) \in E \text{ and } d(x) = \Delta\}| \leq d_H(v) + |C| < 3\sqrt{3}\Delta^{159/160}$. Because G is a Vizing graph, $d(u) > \Delta - 3\sqrt{3}\Delta^{159/160}$ for every neighbor of v. Furthermore, $d_{B-H-C}(v) > \frac{1}{2}\Delta - 3\sqrt{3}\Delta^{159/160}$. So, we can choose some neighbor u of v in $B - H - C$ such that $d_{V-H}(u) > d(u) - \sqrt{3}\Delta^{159/160} > \Delta - 4\sqrt{3}\Delta^{159/160} > \Delta - 8\Delta^{159/160}$. This implies $|B \cup S - H| > \Delta - 8\Delta^{159/160}$.

□

We now move to further refine the statement of the main theorem. We introduce another definition that we will find useful:

Definition 6 *A partition (B, S) of the vertices of a graph $G = (B \cup S, E)$ of maximum degree Δ is called* **weakly Vizing** *if:*

1. $d_B(v) + d_{B-v}(u) \geq \Delta$ *for all* $u, v \in B$,

2. $d_B(u) \leq \frac{1}{2}\Delta$ *for all* $u \in S$,

3. $d_B(v) \geq \frac{1}{2}\Delta$ *for all* $v \in B$.

We will call a graph **weakly Vizing** if there is a weakly Vizing partition of the vertices in G. For example, a Vizing graph $G = (B \cup S, E)$ is always weakly Vizing with the obvious partition (B, S), since $d_B(v) > \frac{1}{2}\Delta$ for all $v \in B$ and $d_B(u) \leq \frac{1}{2}\Delta$ for all $u \in S$. Note that, in a weakly Vizing graph $G = (B \cup S, E)$, the graph induced by B is itself a weakly Vizing graph, with the weakly Vizing partition $(B, \emptyset)$.

An overfull subgraph F of maximum degree Δ in a weakly Vizing graph $G = (B \cup S, E)$ is called **trivial** if F has maximum degree Δ, and $F = B$, $F = B - v$ for some $v \in B$ or $F = B + u$ for some $u \in S$. We will use different versions of the following technical lemma several times throughout this proof:

Lemma 17 (trivial lemma) *Let $G = (B \cup S, E)$ be a weakly Vizing graph of maximum degree Δ. If G contains an overfull subgraph F of maximum degree Δ then one of the following is true:*

(i) *G contains a forbidden subgraph of type 2.*

(ii) *G contains a trivial overfull subgraph.*

Proof: We first remark that the set $R = \{v \in F : d_F(v) < \Delta - \sqrt{\Delta}\}$ is smaller than $\sqrt{\Delta}$. Thus the subgraph H induced by $F - R$ in B has minimum degree greater than $\Delta - 2\sqrt{\Delta}$.

Consider the set $C = \{v \in B - F : d_F > \sqrt{\Delta}\}$. Clearly, $|C| \leq \sqrt{\Delta}$. If $B - F - C$ contains two vertices u and v, then at least one of them, say v, has at least $\frac{1}{2}\Delta$ neighbors in B since $d_B(v) + d_{B-v}(u) \geq \Delta$. Then, $d_{B-F}(v) \geq \frac{1}{2}\Delta - \sqrt{\Delta}$ and, furthermore, H is a forbidden subgraph of type 2. On the other hand, if $|B - F| \leq |C| + 1$, it follows that $|B - F| \leq 2$, since $|E(F, B - F)| \leq \Delta$ and because $d_B(v) + d_{B-v}(u) \geq \Delta$ for all $u, v \in B$ (G is weakly Vizing).

If $B - F = \{u, v\}$, then $|E(F + u + v)| = |E(F)| + d_B(v) + d_{B-v}(u) > \frac{1}{2}(|V(F)| - 1)\Delta + \Delta = \frac{1}{2}(|F + u + v| - 1)\Delta$ and $B = F + u + v$ is an overfull subgraph of G as well. So, in any case, there exists an overfull subgraph F' of maximum degree Δ such that $|B - F'| \leq 1$. Suppose now that $|F' \cap S| \geq 2$, and let u and v be two vertices of $F' \cap S$. Then $|E(F')| = |E(F' - u - v)| + d_F(u) + d_F(v) \leq \frac{1}{2}\Delta(|F'| - 2)$, which contradicts the fact that F' is overfull. So, $|F' \cap S| \leq 1$. Finally, we show that if $|B - F'| = 1$ and $|S \cap F'| = 1$, then B itself is overfull, implying that G contains a trivial overfull subgraph. Suppose that $F' = B - v + u$ for some $v \in B$ and $u \in S$. Then $|E(F')| = |E(F' - u)| + d_F(u) = |E(B)| - d_B(v) + d_F(u) \leq |E(B)|$ and B is overfull. □

Using the above results we can now give a more precise formulation of the main theorem.

Theorem 18 (final refinement) *There exists Δ_0 such that for all Vizing graphs $G = (B \cup S, E)$ of maximum degree $\Delta \geq \Delta_0$ and $|B \cup S| \leq 6\Delta$, one of the following holds:*

(i) *B contains a forbidden subgraph,*

(ii) *G contains a trivial overfull subgraph,*

(iii) *G is Δ edge colorable.*

Furthermore, there is a procedure which runs in $O(2^{|V|})$ time that will output either a Δ edge coloring of an input graph G, a trivial overfull subgraph of G or a forbidden subgraph of B.

The remarks in this section suggest that we are really interested in coloring $E(B)$ and that the edges to S are somewhat superfluous. This motivates the following definitions, and our focussing on B in the proof of theorem 18:

Definition 7 *Let $G = (B \cup S, E)$ be a weakly Vizing graph of maximum degree Δ.*
The **deficiency** *of a vertex $v \in B$ is $def(v) = \Delta - d_B(v)$.*
The **real deficiency** *of a vertex $v \in B$ is $def_r(v) = \Delta - d_G(v)$.*
The **fake deficiency** *of a vertex $v \in B$ is $def_S(v) = d_S(v) = def(v) - def_r(v)$.*
More specifically, the fake deficiency of v with respect to $u \in S$ is $def_u(v) = \mu(u, v)$.

We find it convenient to use the following notation. If $H \subset B$ then $\mathrm{def}(H) = \sum_{v \in H} \mathrm{def}(v)$, $\mathrm{def}_r(H) = \sum_{v \in H} \mathrm{def}_r(v)$ and $\mathrm{def}_S(H) = \sum_{v \in H} \mathrm{def}_S(v)$. If $H \subset B$ and $H' \subset S$ then $\mathrm{def}_{H'}(H) = \sum_{u \in H'} \mathrm{def}_u(H) = \sum_{u \in H'} \sum_{v \in H} \mathrm{def}_u(v)$. Note that $d(S) = \sum_{u \in S} d(S) = \mathrm{def}_S(B)$.

We finish this section with a useful lemma in which we give the necessary and sufficient conditions, in terms of deficiency, for a Vizing graph $G = (B \cup S, E)$ to contain a trivial overfull subgraph:

Lemma 19 *If $G = (B \cup S, E)$ is a Vizing graph then:*

(i) *B is a trivial overfull subgraph if and only if $|B|$ is odd and $def(B) < \Delta$.*

(ii) *$B - v$ is a trivial overfull subgraph for some v in B if and only if $|B|$ is even and $def(B) < 2def(v)$.*

(iii) *$B + u$ is a trivial overfull subgraph for some u in S if and only if $|B|$ is even and $def(B) < 2def_u(B)$.*

Proof: B is a trivial overfull subgraph if and only if $|B|$ is odd and $2|E(B)| > \Delta(|B| - 1)$. The inequality $2E(B) > \Delta(|B| - 1)$ is equivalent to $\mathrm{def}(B) < \Delta$, since $2|E(B)| = \sum_{v \in B} d_B(v) = \sum_{v \in B}(\Delta - \mathrm{def}(v)) = \Delta|B| - \mathrm{def}(B)$.

$B - v$ is a trivial overfull subgraph, for some v in B, if and only if $|B|$ is even and $2|E(B - v)| > \Delta(|B| - 2)$. This inequality is equivalent to $\mathrm{def}(B) < 2\mathrm{def}(v)$, since $2|E(B - v)| = 2|E(B)| - 2d_B(v) = \Delta|B| - \mathrm{def}(B) - 2d_B(v)$.

$B + u$ is a trivial overfull subgraph, for some u in S, if and only if $|B|$ is even and $2|E(B + u)| > \Delta|B|$. This inequality is equivalent to $\mathrm{def}(B) < 2\mathrm{def}_u(B)$, since $2|E(B + u)| = 2|E(B)| + 2d_B(u) = \Delta|B| - \mathrm{def}(B) + 2\mathrm{def}_u(B)$. $\square$

3.2 The edge coloring algorithms

In chapters 4, 5 and 6, we will present several algorithms that attempt to color with Δ colors the edges of Vizing graph $G = (B \cup S, E)$. We assume that the maximum degree of G, Δ, is large, $|B \cup S| \leq 6\Delta$, that no trivial subgraph of G is overfull and that a split partition $(B_1 \cup S_1, B_2 \cup S_2)$ of $B \cup S$ is provided. In case our algorithms fail, we will show the existence of and construct a fail pair (X, Y) in $(B_1 \cup S_1, B_2 \cup S_2)$. We will use this fail pair to construct a forbidden subgraph in B, by applying the forbidden subgraph construction procedure. The main theorem then follows from the proofs of correctness of the edge coloring algorithms.

The main idea of our edge-coloring algorithms is an attempt to construct disjoint matchings $M_1, ..., M_k$ such that $H = G - \cup_{i=1}^{k} M_i$ is a reduction of G that is easily $\Delta(H)$ edge colorable. In most cases, H will be a subgraph of $(B_1 \cup S_1, B_2 \cup S_2)$, i.e. a bipartite subgraph whose edges we can color with $\Delta(H)$ colors using a polynomial time algorithm derived from the proof of Konig's theorem. In a few special cases, H will be a near-bipartite subgraph with no overfull subgraph of maximum degree Δ such that $H - v$ is a subgraph of $(B_1 \cup S_1, B_2 \cup S_2)$ for some $v \in B$. In that case, we can use Reed's polynomial time algorithm, described in appendix A to color the edges of H with $\Delta(H)$ colors. Once we have colored H, we assign the remaining k colors to the matchings $M_1, ..., M_k$.

Our main goal, then, is to construct the matchings $M_1, ..., M_k$. We will do that through two coloring passes, as we did in the regular case.

3.2.1 The two coloring passes and the marking

In the **first coloring pass**, we construct the first Δ_1 matchings $M_1, ..., M_{\Delta_1}$ such that $F = G - \cup_{i=1}^{\Delta_1} M_i$ is a reduction of G, where $\Delta_1 = \frac{1}{2}\Delta + o(\Delta)$.

We will start with an **initial coloring**, i.e. disjoint matchings $M'_1, ..., M'_{\Delta_1}$ such that $\cup_{i=1}^{\Delta_1} M'_i = E(B_1 \cup S_1) \cup E(B_2 \cup S_2)$, as in the regular case. We will end by recursively augmenting each M'_i along vertex disjoint patches constructed in $F = G - M_1 - ... - M_{i-1}$ between vertices of "large" degree (**patching**), thereby obtaining M_i that hits every "large" degree vertex and finally insuring that $F = G - M_1 - M_2 - ... - M_{\Delta_1}$ is a reduction of G. The edges in $B_1 \cup S_1$ and in $B_2 \cup S_2$ that we uncolor while augmenting the patches are added to **reject** graphs R_1 and R_2 that we will color in the reject coloring pass, using disjoint matchings $M_{\Delta_1+1}, ..., M_{\Delta_1+\Delta_2}$, where $\Delta_2 = o(\Delta)$. If we are unable to construct a patch then we will find a fail pair (X, Y).

In order to describe what vertices, missed by an initial matching M_i', have "large" degree and must be patched, we define a **marking** that indicates, for every $i = 1, ..., \Delta_1 + \Delta_2$, which big vertices will be missed by M_i. More precisely a marking of the vertices in B in the matchings $M_1, ..., M_{\Delta_1+\Delta_2}$ is an assignment $m : B \times \{1, 2, ..., \Delta_1 + \Delta_2\} \to \{0, 1\}$ such that $m(v, i) = 1$ if and only if v is missed by the big edges of M_i (recall that an edge (x, y) is big if $x, y \in B$). If $m(v, i) = 1$ we will say that v is marked in M_i or that M_i contains a mark on v. Intuitively each mark on a vertex v represents a "unit" of deficiency. Since we use two types of deficiency, we additionally specify whether a mark is real and M_i misses v, (denoted by $m_r(v, i) = 1$), or whether a mark is fake and M_i hits v with an edge $(v, u) \in (B, S)$ (denoted by $m_u(v, i) = 1$). We call a marking **proper** over an initial coloring $M_1', ..., M_{\Delta_1}'$ if $m_r(v, i) = 1$ implies that M_i' misses v and $m_u(v, i) = 1$ if and only if $(u, v) \in M_i'$.

We define some notation we will find useful. Let $m^{\Delta_1}(v) = \sum_{i=1}^{\Delta_1} m(v, i)$ and $m(v) = \sum_{i=1}^{\Delta_1+\Delta_2} m(v, i)$. We similarly define $m_u^{\Delta_1}(v)$ and $m_u(v)$, where $u \in S$ or $u = r$. If $m^{\Delta_1}(v) = k$ then we will say that v has k marks in the first Δ_1 matchings. For any $A \subseteq B$ we define $m(A) = \sum_{v \in A} m(v)$ and we say that A has $m(A)$ marks; we similarly define $m^{\Delta_1}(A)$, $m_u(A)$ and $m_u^{\Delta_1}(A)$, where $u \in S$ or $u = r$. For any $A \subset S$, we define $m_A(v) = \sum_{u \in A} m_u(v)$ and we similarly define $m_A^{\Delta_1}(v)$ and $m_A(A')$ and $m_{A'}^{\Delta_1}(A')$ where $A' \subset B$. We make the important observations that $H = G - M_1 - ... - M_{\Delta_1+\Delta_2}$ is a reduction of G if and only if $0 \le m_r(v) \le \mathrm{def}_r(v)$ for every $v \in B$ and H is bipartite with edges in $E(B_1 \cup S_1, B_2 \cup S_2)$ only if $m_u(v) = \mathrm{def}_u(v)$ for every $(v, u) \in E(B_1, S_1) \cup E(B_2 \cup S_2)$.

The marking definition turns out to be the heart of the difficulties of our edge coloring algorithms. After defining an initial marking that is proper over the initial matchings $M_1', ..., M_{\Delta_1}$, we will modify the marking and the matchings to "prepare" them for patching. Because of the difficulties involved, we will develop different methods to modify the marking depending on the total deficiency of the input graph ($\mathrm{def}(B)$). This is why we choose to present separate edge-coloring algorithms for graphs of large, medium, small and smallest deficiency (to be precisely defined later).

In the **reject coloring pass**, we color the remaining, uncolored edges in

$B_1 \cup S_1$ and $B_2 \cup S_2$, i.e. the graphs R_1 and R_2, respectively. We start by coloring the edges in $E(R_1)$ and $E(R_2)$ with $\Delta_2 = o(\Delta)$ colors to obtain the **initial coloring** defined by the initial matchings $M'_{\Delta_1+1}, ..., M'_{\Delta_1+\Delta_2}$. We then define a **marking**, that is proper over the initial matchings, and then we **patch** the unmarked vertices missed by each M'_{Δ_1+i} to obtain M_{Δ_1+i}. We insist however that all patches consist of one edge only, as in the regular case. If we fail in augmenting a matching M'_{Δ_1+i} we will prove the existence of (and construct) a fail pair (X, Y).

In the following 2 sections, we describe procedures that equalize the number of big edges in the initial matchings $M'_1, ..., M'_{\Delta_1}$ and $M'_{\Delta_1+1}, ..., M'_{\Delta_1+\Delta_2}$ and that equalize the number of marks in each matching $M_1, ..., M_{\Delta_1+\Delta_2}$. We will find these procedures extremly useful.

3.2.2 Balancing the matchings

Let H be a graph whose vertices are partitioned into sets B and S such that S induces an independent set in H (i.e. $E(H) = E(B) \cup E(B, S)$). We call disjoint matchings $M_1, ..., M_k$ covering $E(H)$ (i.e. $\cup_{i=1}^k M_i = E(H)$) **balanced in B** if for any $1 \leq i < j \leq k$:

$$0 \leq |M_j \cap E(B)| - |M_i \cap E(B)| \leq 1 \tag{3.1}$$

We can modify any disjoint matchings $M_1, ..., M_k$ covering $E(H)$ so that inequalities 3.1 are satisfied for all $1 \leq i < j \leq k$ by recursively repeating the following procedure:

The balancing procedure

> We pick matchings M_i and M_j with $i < j$ such that either **(a)** $|M_j \cap E(B)| - |M_i \cap E(B)| > 1$ (see figure 2.1) or **(b)** $|M_j \cap E(B)| - |M_i \cap E(B)| < 0$. We consider the connected components of $M_i \cup M_j$ consisting of cycles and paths whose edges alternate between edges in M_i and edges in M_j. We observe that in all alternating cycle components, the number of big edges (whose endpoints are in B) is even and half of the big edges belong to M_i while the other half belongs to M_j. An alternating path, however, may contain an odd number of big edges. So, in case **(a)**, there must exist a connected component of $M_i \cup M_j$ that is an alternating path P with one more big edge in M_j than in M_i (see figure 2.1). In case **(b)**, there must exist a connected component

of $M_i \cup M_j$ that is an alternating path P with one more big edge in M_i than in M_j. In both cases, we switch the color of each edge of P.

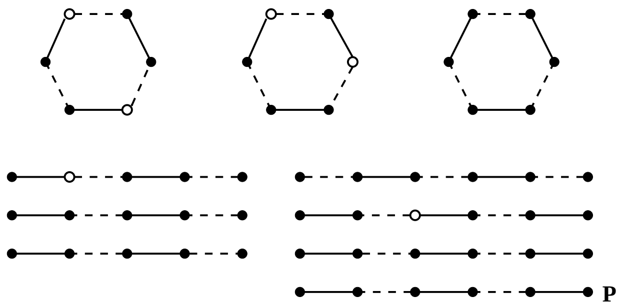

Figure 3.1: M_j (full edges) and M_i (dashed edges) with $|M_j \cap E(B)| - |M_i \cap E(B)| = 2$, with big and small vertices represented with dots and circles, respectively

It is easy to check that $M_1, ..., M_k$ are balanced after at most $O(n^2)$ iterations.

3.2.3 Equalizing the marking

Let H be a graph whose vertices are partitioned into sets B and S such that the vertices in S induce an independent set in H (i.e. $E(H) = E(B) \cup E(B, S)$). Let $M_1, ..., M_k$ be disjoint matchings covering $E(H)$ and balanced in B. We call a proper marking m of the big vertices of B in the matchings $M_1, ..., M_k$ **equalized** if for any $1 \leq i < j \leq k$:

$$0 \leq m(B, i) - m(B, j) \leq 2 \tag{3.2}$$

where $m(B, i) = \sum_{v \in B} m(v, i) = 1$ for every $i = 1, ..., k$. We can modify a marking m and the matchings $M_1, ..., M_k$ so that inequalities 3.2 are satisfied, while maintaining the properties that $M_1, ..., M_k$ are disjoint, cover $E(H)$ and are balanced in B, by recursively repeating the following:

Equalizing the marking We pick two matchings M_i and M_j with $i < j$ such that either **(a)** $m(B, i) - m(B, j) \geq 3$ (see figure 3.2) or **(b)** $m(B, j) - m(B, i) \geq 1$. We consider the connected components of $M_i \cup M_j$, which consist of alternating cycles and alternating paths (which may consist of a single vertex). We remark that in any alternating cycle component, the number of big vertices marked in M_i is equal to the number of vertices marked in M_j. On the other hand, an alternating path component may contain up to two more vertices marked in M_i or in M_j. Depending on the case, we do as follows

(a) there must exist either an alternating path P with an even (may be zero) number of edges in $E(B)$ and with $k = 1$ more marks in M_i than in M_j (see figure 3.2), or a pair of alternating paths P_1 and P_2, each with an odd number of edges in $E(B)$ such that P_1 has one more edge in $M_i \cap E(B)$ and P_2 has one more edge in $M_j \cap E(B)$ and so that $P_1 \cup P_2$ has $k = 1$ or $k = 2$ more marks in M_i than in M_j.

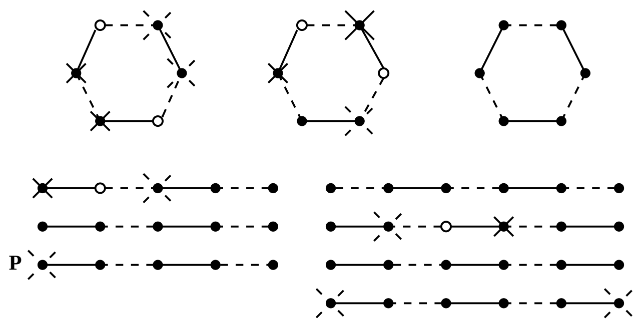

Figure 3.2: M_j (full edges) and M_i (dashed edges) with a marking on the vertices (full for marks in M_j and dashed for marks in M_i) that satisfies $m(B,i) - m(B,j) \geq 3$

(b) there must exist either an alternating path P with an even (may be zero) number of edges in $E(B)$ and with $k = 1$ more marks in M_j than in M_i or a pair of alternating paths P_1 and P_2, each with an odd number of edges in $E(B)$ such that P_1 has one more edge in $M_i \cap E(B)$ and P_2 has one more edge in $M_j \cap E(B)$ and so that $P_1 \cup P_2$ has $k = 1$ or $k = 2$ more marks in M_j than in M_i.

In both cases we pick the path P or the pair of paths P_1, P_2, whichever exists and we switch the color of the edges on the path(s) and the marks on the vertices of the path(s) (i.e. a vertex marked in M_i becomes marked in M_j and vice versa). Thus we decrease the difference between the number of marks in M_i and M_j by k.

Chapter 4

The large and medium deficiency cases

In this chapter, we present two algorithms that attempt to construct disjoint matchings $M_1, ..., M_k$ such that $H = G = M_1 - ... - M_k$ is a bipartite reduction of a Vizing graph $G = (B \cup S, E)$ of deficiency at least $\Delta^{12/10}$. We will assume that G contains no trivial overfull subgraphs, that the maximum degree Δ of G is large enough, that $|B \cup S| \le 6\Delta$ and that a split partition $(B_1 \cup S_1, B_2 \cup S_2)$ of $B \cup S$ edge is provided. In case our algorithms fail, we will show the existence of and construct a fail pair (X, Y) in $(B_1 \cup S_1, B_2 \cup S_2)$.

4.1 The large deficiency case

We consider first the large deficiency graphs, which we define as graphs with more than $2\Delta^{9/10}$ vertices of deficiency greater than $\Delta^{9/10}$. In other words, if we define $B^- = \{v \in B : \mathrm{def}(v) > \Delta^{9/10}\}$ then $|B^-| > 2\Delta^{9/10}$. Note that if $G = (B \cup S, E)$ has large deficiency then $\mathrm{def}(B) > 2\Delta^{18/10}$.

4.1.1 The first coloring pass

In the first pass, we construct disjoint matchings $M_1, ..., M_{\Delta_1}$, where $\Delta_1 = \frac{1}{2}\Delta + \delta = \lceil \frac{1}{2}\Delta + \Delta^{11/20} \rceil$, such that $F = G - M_1 - ... - M_{\Delta_1}$ is a reduction of G and $\cup_{i=1}^{\Delta_1} M_i$ contains all edges in $E(B_1 \cup S_1) - E(R_1)$ and in $E(B_2 \cup S_2) -$

$E(R_2)$ where R_1 and R_2 are reject subgraphs in $B_1 \cup S_1$ and in $B_2 \cup S_2$, respectively, of maximum degree less than $\frac{1}{4}\Delta^{9/10}$ such that $|E(R_1)|$ and $|E(R_2)|$ contain less than $\Delta^{17/10}$ edges. In addition, we insist that the marking satsfies

$$|m^{\Delta_1}(v) - \frac{1}{2}\mathrm{def}(v)| < \frac{1}{4}\Delta^{9/10} \text{ for all } v \in B \tag{4.1}$$

To help us with the marking, we find it useful to partition the vertices of B as follows:

$$\begin{aligned} B^l &= \{x \in B : \mathrm{def}_r(x) \leq 4\Delta^{11/20}\} \\ B^s &= \{x \in B : \mathrm{def}_r(x) \geq 2\Delta^{9/10}\} \end{aligned}$$

Thus, $B - B^l - B^s$ is the set of vertices with real deficiency between $4\Delta^{11/20}$ and $2\Delta^{9/10}$. We remark that B^s may be empty and, furthermore, we could have $\mathrm{def}_r(v) = 0$) for all $v \in B$!

The initial coloring

We initially construct matchings $M'_1, ..., M'_{\Delta_1}$, balanced in B_1 and in B_2, such that $\cup_{i=1}^{\Delta_1} M'_i = E(B_1 \cup S_1) \cup E(B_2 \cup S_2)$ as follows:

1. We apply Fournier's algorithm to color the edges of $E(B_1 \cup S_1)$ with Δ_1 colors to obtain the disjoint matchings $M^1_1, ..., M^1_{\Delta_1}$. We balance these matchings in B_1 using the balancing procedure from section 3.2.2. We similarly construct $M^2_1, ..., M^2_{\Delta_1}$, balanced in B_2. For every $i = 1, ..., \Delta_1$, we set $M'_i = M^1_i \cup M^2_i$.

Since $0 \leq c_B = |E(B_1)| - |E(B_2)| < \frac{1}{2}\Delta$, it follows that $0 \leq |M^1_i - M^2_i| \leq 1$ for all $i = 1, ..., \Delta_1$; actually, $|M^1_i - M^2_i| = 1$ for exactly c_B indices i. Note that $\Delta_1 - d_{B_1 \cup S_1}(v)$ of the matchings miss $v \in B_1$. Using the equivalent fact about $v \in B_2$ and the properties of a split partition, it follows that the number of matchings missing $v \in B$ is at least $\frac{1}{2}\mathrm{def}_r(v) + \frac{3}{4}\delta$ and at most $\frac{1}{2}\mathrm{def}_r(v) + \frac{5}{4}\delta$.

The initial marking

We now define an initial marking of the vertices in B that is proper over the matchings $M'_1, ..., M'_{\Delta_1}$ and balanced in B_1 and B_2:

2.1 We set $m_u(v, i) = 1$ for every $v \in B$, every $u \in S$ and every $i = 1, ..., \Delta_1$ such that $(v, u) \in M'_i$. Then, for every $v \in B - B^l$, we set $m_r(v, i) = 1$

for every M_i' missing v. Finally, for every $v \in B^l$, we pick a set $\mathcal{R}$ of $\lceil \frac{1}{2}\text{def}_r(v) \rceil$ matchings (among $M_1', ..., M_{\Delta_1}'$) missing v. We set $m_r(v,i) = 1$ for every $M_i' \in \mathcal{R}$.

2.2 We equalize the marking defined in **2.1**, separately in B_1 and B_2, over the initial matchings $M_1', ..., M_{\Delta_1}'$ using the equalizing procedure from 3.2.3.

We observe that, even though we modify $M_1', ..., M_{\Delta_1}'$ in **2.2**, the matchings remain balanced over B_1 and B_2 and still contain all edges in $E(B_1 \cup S_1)$ and in $E(B_2 \cup S_2)$. The marking is valid since $0 \leq m_r(v) \leq \text{def}_r(v)$ for all $v \in B$. Furthermore,

$$|m^{\Delta_1}(v) - \frac{1}{2}\text{def}(v)| < 2\delta \text{ for all } v \in B \tag{4.2}$$

from which it follows that $|m^{\Delta_1}(B_1) - \frac{1}{2}\text{def}(B_1)| < 2|B_1|\delta$. Equivalent results hold for B_2. Let $n(B_1, i)$ and $n(B_2, i)$ be the numbers of unmarked vertices in B_1 and in B_2, respectively, missed by M_i'.

Claim 9 *For all* $i = 1, .., \Delta_1$*:*

$m(B_1, i), m(B_2, i) > \frac{1}{2}\Delta^{4/5}$, *and*

$n(B_1, i), n(B_2, i) < 8\delta$.

Proof: We prove both statements only for B_1, as the corresponding results for B_2 follows by a symmetric argument. We note that $m^{\Delta_1}(B_1) > \frac{1}{2}\text{def}(B_1) - 2|B_1|\delta > \frac{1}{2}\Delta^{18/10} - \Delta^{16/10}$. Since the marking is equalized in B_1, it follows that $m(B_1, i) > \frac{1}{\Delta_1}m^{\Delta_1}(B_1) - 2 > \frac{1}{2}\Delta^{4/5}$.

Only vertices in B^l can be unmarked in an missed by a matching M_i'. Since $v \in B^l$ is marked in $\lceil \frac{1}{2}\text{def}_r(v) \rceil$ matchings, v is unmarked in and missed by at most $\frac{5}{4}\delta$ matchings. Since the marking is equalized in B_1 and the markings are balanced in B_1, any M_i' misses at most $\frac{1}{\Delta_1}\frac{5}{4}\delta|B_1| + 2 < 8\delta$ vertices in B_1. $\square$

The patching

For $i = 1, ..., \Delta_1$, we recursively construct M_i by augmenting M_i' along vertex disjoint patches we construct in $F = G - R_1 - R_2 - M_1 - M_2 - ... - M_{i-1}$.

We construct the patches in F so that every unmarked big vertex v (i.e. $m(v,i)=0$) missed by M_i' is included in some patch; by augmenting the patches we will insure that these vertices are hit by M_i. Note that only vertices in B^l can be unmarked in and missed by an initial matching M_i'. Each augmentation leaves uncolored edges that we add to the reject graphs R_1 and R_2. If we fail to construct a patch then we return a fail pair (X,Y) in $(B_1 \cup S_1, B_2 \cup S_2)$.

In order to describe the construction of the disjoint patches in $F = G - R_1 - R_2 - M_1 - ... - M_{i-1}$, we first define some terminology. In this large deficiency case, a patch is either a sequence of vertices $x^0, y^0, x^1, y^1, ..., x^j, y^j$ or $x^0, y^0, x^1, y^1, ..., x^j, y^j, x^j+1$ such that x^0 is an unmarked vertex in B_1 or in B_2 missed by M_i', $(x^l, y^l) \in E(F) \cap E(B_1 \cup S_1, B_2 \cup S_2)$ and $(y^l, x^{l+1}) \in M_i'$ for $l = 0, ..., j$. We call y^j and x^{j+1} the **external** vertices of the patch.

We construct the patch P starting at some unmarked vertex $x \in B_l$ missed by M_i' and not included in any of the already constructed patches in F as follows:

3.1 We first define unavailable and usable vertices. We call $v \in B \cup S$ **internally unavailable** if v is belongs to any of the patches already constructed in F or to any of the patches constructed for one of the previous $8\lceil \Delta^{1/10} \rceil$ matchings $(M_{i-1}, ..., M_{i-8\lceil \Delta^{1/10} \rceil})$. We call $v \in B \cup S$ **externally unavailable** if v is an external vertex of any of the patches already constructed in F or any of the patches constructed for one of the previous $8\lceil \Delta^{1/10} \rceil$ matchings. We call $v \in B \cup S$ **usable** if v is unmarked in and missed by M_i', if v is missed by M_i' and v is not externally unavailable, or if $(v,u) \in M_i'$ and neither u nor v is externally unavailable. Finally, we call $v \in B \cup S$ **internally usable** if $(v,u) \in M_i'$ and neither u nor v is internally unavailable.

We observe that all internally usable vertices are usable.

3.2 We recursively build the sets $X^0, ..., X^{k+1}$ and $Y^0, ..., Y^k$ where $k = 6\lceil \Delta^{1/20} \rceil$ as follows:

$X^0 = \{x\}$, and for $l = 0, ..., k$,

$Y^l = \{v \in B : v$ is usable and $\exists u \in X^l$ such that $(u,v) \in E(F) \cap E(B_1 \cup S_1, B_2 \cup S_2)\}$.

$X^{l+1} = \{v \in B - B^s : \exists u \in Y^l$ s.t. $(u,v) \in M_i'$ and u, v are internally usable$\}$

$Y^l \geq X^{l+1}$ for every $l = 0, ..., k$, and $Y^j > X^{j+1}$ if and only if there is a usable vertex $y \in Y^j$ that is either missed by M_i' or hit with an edge $(y, z) \in M_i'$ such that $z \in B^s \cup S$ and z is usable.

3.3 If for some j, there is a vertex $y \in Y^j$ that is missed by M_i', we construct the patch P defined by the sequence of vertices $x = x^0, y^0, x^1, y^1, ...,$ $y^{j-1}x^j, y^j = y$ where $x^l \in X^l$, $y^l \in Y^l$ and $(x^l, y^l) \in E(F) \cap E(B_1 \cup S_2, B_2 \cup S_2)$, for $l = 0, ..., j$, and $(y^l, x^{l+1}) \in M_i'$ for $l = 0, ..., j-1$. In addition we adjust the marking as follows:

i. If $y \in B$ and $m_r(y, i) = 1$, we reset it to $m_r(y, i) = 0$;

ii. If $y \in S$, we set $m_y(x^j, i) = 1$;

Finally, for every small vertex y^l that appears on P such that $0 \leq l \leq j-1$, we reset $m_{y^l}(x^{l+1}, i) = 0$ and we set $m_{y^l}(x^l, i) = 1$.

3.4 If for some j, there is a vertex $y \in Y^j$ that is hit with an edge $(y, z) \in M_i'$ such that $z \in B^s \cup S$ (so that y and z are usable), we construct the patch P defined by the sequence of vertices $x = x^0, y^0, x^1, y^1, ..., y^{j-1}$, x^j, y^j, z as above. In addition, we adjust the marking as follows:

iii. If $y \in B$ and $z \in S$, we reset $m_z(y, i) = 0$;

iv. If $y \in B$ and $z \in B^s$, we set $m_r(z, i) = 1$;

v. If $y \in S$ and $z \in B^s$, we reset $m_y(z, i) = 0$ and we set $m_r(z, i) = 1$ and $m_y(x^j, i) = 1$.

Finally, for every small vertex y^l that appears on p such that $0 \leq l \leq j-1$, we reset $m_{y^l}(x^{l+1}, i) = 0$ and we set $m_{y^l}(x^l, i) = 1$.

If we are successful in constructing all the patches in every initial matching M_i', we obtain disjoint matchings $M_1, ..., M_{\Delta_1}$ such that $\cup_{i=1}^{\Delta_1} M_i = (E(B_1 \cup S_1) - E(R_1)) \cup E(B_2 \cup S_2) - E(R_2)$. We now show that the marking and the reject graphs R_1 and R_2 satisfy the desired properties.

A vertex $v \in B \cup S$ is an external vertex in at most $\frac{\Delta_1}{8\lceil \Delta^{1/10} \rceil}$ patches and an internal vertex in at most $\frac{\Delta_1}{8\lceil \Delta^{1/10} \rceil}$ additional patches. So at most $\frac{\Delta_1}{4\lceil \Delta^{1/10} \rceil} < \frac{1}{4}\Delta^{9/10}$ edges incident to v are rejected. Furthermore, since there are at most 16δ unmarked vertices missed in each M_i', $|E(R_1)| + |E(R_2)| < \Delta_1 16\delta 6\Delta^{1/20} < \Delta^{17/10}$.

Note that we add or delete less than $\frac{\Delta_1}{4\lceil\Delta^{1/10}\rceil} < \frac{1}{4}\Delta^{9/10} - 2\delta$ real or fake marks to any vertex v, in addition to the marks already assigned to v in the marking step. It follows that $|m^{\Delta_1}(v) - \frac{1}{2}\text{def}(v)| < \frac{1}{4}\Delta^{9/10}$ for every $v \in B$, satisfying 4.1. Because we never add a real mark to any $v \in B - B^s$ during the patching, it follows that $0 \leq m_r(v) \leq \text{def}_r(v)$. Now, $v \in B^s$ was initially marked in at least $\frac{1}{2}\text{def}_r(v) + \frac{3}{4}\delta$ and at most $\frac{1}{2}\text{def}_r(v) + \frac{5}{4}\delta$ initial matchings. Thus, $|m_r^{\Delta_1}(v) - \frac{1}{2}\text{def}_r(v)| < \frac{1}{2}\Delta^{9/10}$, and since $\text{def}_r(v) > 2\Delta^{9/10}$, it follows that $0 \leq m_r(v) \leq \text{def}_r(v)$. The marking is then a valid one.

If we fail to construct a patch:

3.5 If $X^{l+1} = Y^l$ for every $l = 0, ..., 6\lceil\Delta^{1/20}\rceil$, we pick the smallest $j \geq 1$ such that $|Y^j| \leq |X^j| + \frac{1}{2}\Delta^{19/20}$ (we prove in claim 10 below that this inequality must hold for some j). Let F_1 and F_2 be vertices in B_1 and B_2, respectively, that are *not* usable and let $X = X^j$ and $Y = Y^j \cup F_1$, if $Y^j \subset B_1 \cup S_1$, or $Y = Y^j \cup F_2$, if $Y^j \subset B_2 \cup S_2$.

Claim 10 *(X, Y) forms a fail pair in $(B_1 \cup S_1, B_2 \cup S_2)$.*

Before we prove the claim, we give lower bounds on the degrees of vertices in F. Since $d^F(v) = d(v) - (\Delta_1 - m_r^{\Delta_1}(v))$, it follows that $|d^F(v) - \frac{1}{2}d(v)| < \frac{1}{2}\Delta^{9/10}$. Furthermore, since $d^F_{B_1 \cup S_1}(v_1) \leq \Delta(R_1) < \frac{1}{4}\Delta^{9/10}$ for every $v_1 \in B_1$, it follows that $d^F_{B_2 \cup S_2}(v_1) > \frac{1}{2}d(v_1) - \Delta^{9/10}$. Equivalently, $d^F_{B_1 \cup S_1}(v_2) > \frac{1}{2}d(v_1) - \Delta^{9/10}$.

Proof: We assume that $x \in B_1 \cap B^l$; a symmetric argument does the job for $x \in B_2 \cap B^l$. Since there are at most 2 external vertices in every patch and there are at most 16δ patches per matching (by claim 9), it follows that the number of externally unavailable vertices in B_1 is at most $8\Delta^{1/10}16\delta$ so that $|F_1| < \Delta^{7/10}$. A symmetrical argument gives $|F_2| < \Delta^{7/10}$.

We note that $|X^1| = |Y^0| \geq d^F_{B_2 \cup S_2}(x) - |F_2| \geq \frac{1}{4}\Delta - 2\Delta^{9/10}$. If we assume that $|Y^l| > |X^l| + \frac{1}{2}\Delta^{19/20}$ for all $1 \leq l \leq 6\lceil\Delta^{1/20}\rceil$, it follows that $|X^{6\lceil\Delta^{1/20}\rceil}| > 3\Delta > |B_1|$, a contradiction. So we must have $|Y^j| \leq |X^j| + \frac{1}{2}\Delta^{19/20}$ for some j between 1 and $6\lceil\Delta^{1/20}\rceil$.

In order to show that the pair (X, Y), as constructed in the procedure forms a fail pair in $(B_1 \cup S_1, B_2 \cup S_2)$, we must show that the following three conditions hold. We prove them only for the case when $X \subset B_1 - B^s$ and $Y \subset B_2 \cup S_2$ (the case $X \subset B_2 - B^s$ and $Y \subset B_1 \cup S_1$ is symmetric):

$|Y| < |X| + \Delta^{19/20}$

Proof: $|Y| = |Y^j| + |F_2| < |X^j| + \frac{1}{2}\Delta^{19/20} + \Delta^{7/10} < |X| + \Delta^{19/20}$.

For all $v \in X$: $d_Y(v) > \frac{1}{2}\Delta - \Delta^{19/20}$.

Proof: For every $v \in X$, $d_Y(v) \geq d_Y^F(v) = d^F_{B_2 \cup S_2}(v) > \frac{1}{2}d(v) - \Delta^{9/10}$. Since every $v \in X$ belongs to $B - B^s$, it follows that $d(v) > \Delta - 2\Delta^{9/10}$ implying $d_Y(v) > \frac{1}{2}\Delta - \Delta^{19/20}$.

$|B_1 - X| > \frac{1}{4}\Delta - \Delta^{19/20}$.

Proof: Since $m(B_2, i) > \frac{1}{2}\Delta^{4/5}$ and $|F_2| < \Delta^{7/10}$, there exists a marked big vertex $v \in B_2 - F_2$, such that v is usable, but $d_X^F(v) = 0$. Thus, $|B_1 - X| \geq d^F_{B_1}(v) > \frac{1}{4}\Delta - \Delta^{19/20}$.

□

4.1.2 The reject coloring pass

We now attempt to construct the remaining disjoint matchings $M_{\Delta_1+1}, ..., M_{\Delta_1+\Delta_2}$, where $\Delta_2 = \lceil \frac{1}{4}\Delta^{19/20} \rceil$, such that $H = F - \cup_{i=1}^{\Delta_2} M_{\Delta_1+i}$ is a bipartite reduction of F (and thus G).

The initial coloring

1. We color the edges in $E(R_1)$ and $E(R_2)$ with Δ_2 colors to obtain matchings $M'_{\Delta_1+1}, ..., M'_{\Delta_1+\Delta_2}$, balanced in B_1 and in B_2, such that

$$\cup_{i=1}^{\Delta_1} M'_{\Delta_1+i} = E(R_1) \cup E(R_2)$$

using Fournier's edge-coloring algorithm and the balancing procedure.

We can apply Fournier's algorithm because $\Delta(R_1), \Delta(R_2) \leq \Delta_2 - 1$. Since $|E(R_1)| + |E(R_2)| < \Delta^{17/10}$ and the initial matchings are balanced in B_1 and in B_2, it follows that $|M'_{\Delta_1+i} \cap E(B)| \leq 4\Delta^{4/5}$ for all $i = 1, ..., \Delta_2$.

The marking

We define an initial marking of the big vertices that is proper over the initial matchings $M'_{\Delta_1+1}, ..., M'_{\Delta_1+\Delta_2}$ as follows:

2. For every $v \in B$, $u \in S$ and $i = 1, ..., \Delta_2$, we set $m_u(v, \Delta_1 + i) = 1$ if $(u, v) \in M'_{\Delta_1+i}$. We then equalize this marking in B_1 and in B_2 using the equalizing procedure.

Since $M'_{\Delta_1+1}, ..., M'_{\Delta_1+\Delta_2}$ are balanced in B_1 and in B_2 and the marking is equalized in B_1 and B_2, it follows that the number of big vertices hit by M'_{Δ_1+i} is $2|M'_{\Delta_1+i} \cap E(B)| + m_S(B, \Delta_1 + i) < 10\Delta^{4/5}$, for all $i = 1, ..., \Delta_2$. We remark that we only put fake marks in this step: we will put additional real marks in each iteration of the patching step.

The patching

For $i = 1, 2, ..., \Delta_2$ we recursively attempt to construct M_{Δ_1+i} by augmenting M'_i with a matching M^* that we construct in $H = F - M_{\Delta_1+1} - ... - M_{\Delta_1+i-1}$. Let $m_r^{\Delta_1+i}(v) = \sum_{j=1}^{\Delta_1+i} m(v, j)$. We attempt to construct M^* as follows:

3.1 Let E_1 and E_2 be the subsets of vertices in B_1 and B_2, respectively, that are hit by M'_{Δ_1+i}, and let U_1 and U_2 be the sets of vertices v in B_1 and B_2 that are missed by M'_i such that $m_r^{\Delta_1+i-1}(v) = \mathrm{def}_r^{\Delta_1+i-1}(v)$ (i.e. vertices with no remaining real deficiency in iteration i). Note that all vertices in $F_1 = B_1 - E_1 - U_1$ and $F = B_2 - E_2 - U_2$ are missed by M'_{Δ_1+i} and have some remaining real deficiency. We set $m_r(v, \Delta_1 + i) = 1$ for every $v \in F_1 \cup F_2$.

Remark: $|E_1|, |E_2| < 10\Delta^{4/5}$.

3.2 We attempt to construct a matching M^* in the bipartite graph $H \cap (U_1 \cup F_1 \cup S_1, U_2 \cup F_2 \cup S_2)$, such that each vertex in $U_1 \cup U_2$ is an endpoint of a matching edge. If successful, we obtain M_{Δ_1+i} by adding M^* to M'_{Δ_1+i}. We adjust the marking so that $m_u(v, \Delta_1 + i) = 1$ for every $v \in U_1 \cup U_2$ such that there is $u \in S$ and $(v, u) \in M^*$; we also reset $m_r(v, \Delta_1 + i) = 0$ for every $v \in F_1 \cup F_2$ for which there exists $u \in U_1 \cup U_2$ and $(v, u) \in M^*$.

If we successfully construct matchings $M_{\Delta_1+1}, ..., M_{\Delta_1+\Delta_2}$ then $H - M_{\Delta_1+1} - ... - M_{\Delta_1+\Delta_2}$ is obviously a bipartite reduction of F.

3.3 If not successful, we either find the sets $X \subset U_1$ and $Y' = N^H_{U_2 \cup F_2 \cup S_2}(X)$ such that $|X| > |Y'|$ and we set $Y = Y' \cup E_2$, or the sets $X \subset U_2$ and $Y' = N^H_{U_1 \cup F_1 \cup S_1}(X)$ such that $|X| > |Y'|$ and we set $Y = Y' \cup E_1$.

Claim 11 *(X, Y) forms a fail pair in $(B_1 \cup S_1, B_2 \cup S_2)$.*

Proof: We assume $X \subset U_1$; the proof for $X \subset U_2$ follows by a symmetric argument. We must show that the following three properties hold:

$|Y| < |X| + \Delta^{19/20}$.
Proof: $|Y| \leq |Y'| + |E_2| < |X| + \Delta^{19/20}$.

$d_Y(v) > \frac{1}{2}\Delta - \Delta^{19/20}$ for all $v \in X$:
Proof: Every $v \in X$ satisfies $d^H(v) = \Delta - \Delta_1 - (i-1)$, so that $d_Y(v) \geq d_Y^H(v) = d^H_{B_2 \cup S_2}(v) = d^H(v) - d^H_{B_1 \cup S_1}(v) > \frac{1}{2}\Delta - \delta - (i-1) - \frac{1}{4}\Delta^{9/10} > \frac{1}{2}\Delta - \Delta^{19/20}$.

$|B_1 - X| > \frac{1}{4}\Delta - \Delta^{19/20}$.
Proof: If there exist $v \in B_2 - Y$, then $d_X^H(v) = 0$ and $d^H_{B_1}(v) > d^F_{B_1}(v) - \Delta_2 > \frac{1}{2}d(v) - 2\Delta^{9/10} > \frac{1}{4}\Delta - \Delta^{19/20}$.

Suppose now that $B_2 \subseteq Y$. Then, $0 \leq |B_1 - X| < |B_1| - |X| < |B_2| - |Y| + 10\Delta^{4/5} \leq 10\Delta^{4/5}$. It also follows that $|Y \cap S_2| < 10\Delta^{4/5}$. Since $|B^-| > 2\Delta^{9/10}$ and because $(B_1 \cup S_1, B_2 \cup S_2)$ is a split partition of $(B \cup S)$, it follows that $|B^- \cap B_1| \geq \Delta^{9/10}$. Clearly there must exist some vertex $v \in X \cap B^-$. Then $d^F_{B_2}(v) < \frac{1}{2}d_B(v) + \frac{1}{8}\delta < \frac{1}{2}\Delta - \Delta^{9/10} + \delta$. Since $v \in X$, however, $d^H_{B_2 \cup S_2}(v) = d^H(v) - d^H_{B_1 \cup S_1}(v) \geq \Delta - \Delta_1 - (i-1) - \frac{1}{4}\Delta^{9/10} > \frac{1}{2}\Delta - \frac{1}{2}\Delta^{9/10} - \delta$. So, $d^H_{S_2}(v) > \frac{1}{2}\Delta^{9/10} - 2\delta$, which implies $|Y \cap S_2| > \frac{1}{2}\Delta^{9/10}$, contradicting $|Y \cap S_2| < 10\Delta^{4/5}$.

□

4.2 The medium deficiency case

We now consider the case of a medium deficiency graph $G = (B \cup S, E)$, which we define as graphs with deficiency $\mathrm{def}(B) \geq \Delta^{12/10}$ but with fewer than $2\Delta^{9/10}$ big vertices of deficiency greater than $\Delta^{9/10}$. In other words the set $B^- = \{v \in B : \mathrm{def}(v) > \Delta^{9/10}\}$ is of size at most $2\Delta^{9/10}$. Note that this implies that $\mathrm{def}(B) < 2\Delta^{9/10}\frac{1}{2}\Delta + 6\Delta\Delta^{9/10} = 7\Delta^{19/10}$.

As $B - B^-$ is "almost" regular and "almost" equal to the whole graph, we will apply essentially the same patching technique we developed in the regular case by insisting that all patches go through $B - B^-$. The difficulty now is what vertices missed by an initial matching to patch. So, we take more care in defining the marking in both coloring passes.

4.2.1 The first coloring pass

We construct, in the first coloring pass, disjoint matchings $M_1, ..., M_{\Delta_1}$ such that $F = G - M_1 - ... - M_{\Delta_1}$ is a reduction of G and $\cup_{i=1}^{\Delta_1} M_i$ contains $E(B_1 \cup S_1) - E(R_1)$ and $E(B_2 \cup S_2) - E(R_2)$ where R_1 and R_2 are reject subgraphs of $B_1 \cup S_1$ and $B_2 \cup S_2$, respectively, of maximum degree less than $\Delta^{9/10}$ such that $|E(R_1) \cap E(B_1)| = |E(R_2) \cap E(B_2)| < \frac{1}{12}\Delta^{19/10}$ and $||E(R_1) \cap E(B_1, S_1)| + |E(R_2) \cap E(B_2, S_2)|| \leq 4\Delta_1$. For technical reasons, we insist that $d_B^F(v) > \frac{1}{2}d_B(v) - \Delta^{9/10}$ which will be true if we require our marking to satisfy

$$|m^{\Delta_1}(v) - \frac{1}{2}\text{def}(v)| < \frac{1}{2}\Delta^{9/10} \tag{4.3}$$

Furthermore, as we need substantial amounts of deficiency remaining for the reject coloring pass, but we also need substantial number of marks, we require that the marking in the first coloring pass satisfies, for $k = 1, 2$,

$$\frac{4}{10}\text{def}(B_k) < m^{\Delta_1}(B_k) < \frac{6}{10}\text{def}(B_k) \tag{4.4}$$

The initial matchings

We first construct the matchings $M'_1, ..., M'_{\Delta_1}$, balanced in B_1 and in B_2, so that $\cup_{i=1}^{\Delta_1} M'_i = E(B_1 \cup S_1) \cup E(B_2 \cup S_2)$. To do this we apply Fournier's algorithm and the balancing procedure from section 3.2.2, exactly as we did in the large deficiency case of section 4.1. We observe that the number of matchings missing $v \in B_1$ is exactly $\Delta_1 - d_{B_1 \cup S_1}(v)$, implying that more than $\frac{1}{2}\text{def}_r(v) + \frac{3}{4}\delta$ matchings miss v, but no more than $\frac{1}{2}\text{def}_r(v) + \frac{5}{4}\delta$. The same is true for $v \in B_2^*$.

The initial marking

We define an initial marking that is proper over modified initial matchings $M'_1, ..., M'_{\Delta_1}$, still balanced in B_1 and B_2 and still satisfying $\cup_{i=1}^{\Delta_1} M'_i = E(B_1 \cup S_1) \cup E(B_2 \cup S_2)$:

2.1 We initially set $m_u(v,i) = 1$ for every $u \in S$, $v \in B$ and i such that $(u,v) \in M'_i$.

2.2 Then, for every $v \in B$, we define a target $t(v)$ equal to $\lceil \frac{1}{2}\mathrm{def}_r(v)\rceil$ or $\lfloor \frac{1}{2}\mathrm{def}_r(v)\rfloor$ so that $t(B_1) = \sum_{v\in B_1} t(v) = \lceil \frac{1}{2}\mathrm{def}_r(B_1)\rceil$ and $t(B_2) = \sum_{v\in B_2} t(v) = \lceil \frac{1}{2}\mathrm{def}_r(B_2)\rceil$.

2.3 For every $v \in B$, we pick a set $\mathcal{R}$ of $t(v)$ matchings (among $M'_1, ..., M'_{\Delta_1}$) missing v and we set $m_r(v,i) = 1$ for every $M'_i \in \mathcal{R}$.

2.4 We equalize the marking in B_1 and B_2 using the equalizing procedure from 3.2.3.

Note that, while $M'_1, ..., M'_{\Delta_1}$ may be modified in step **2.4**, they are still balanced in B_1 and in B_2 and $\cup_{i=1}^{\Delta_1} M'_i = E(B_1 \cup S_1) \cup E(B_2 \cup S_2)$.

Claim 12

$$\frac{4}{10}\mathit{def}(B_k) + 4\Delta_1 < m^{\Delta_1}(B_k) < \frac{6}{10}\mathit{def}(B_k) \tag{4.5}$$

$$\frac{1}{3}\Delta^{1/5} + 4 < m(B_k,i) < \frac{3}{2}\Delta^{9/10} \tag{4.6}$$

for all $i = 1, ..., \Delta_1$ and $k = 1, 2$.

Proof: We only prove the claim for $k = 1$, since a symmetric proof will work for $k = 2$. Let $B_1^* = \{v \in B_1 : \mathrm{def}_S(v) > 400\Delta^{1/10}\}$. We note that $\mathrm{def}_S(B_1 - B_1^*) < |B_1|400\Delta^{1/10} < 1200\Delta^{11/10}$. We also note that $\frac{9}{20}\mathrm{def}_S(v) < \frac{1}{2}\mathrm{def}_S(v) - \frac{1}{8}\sqrt{\mathrm{def}_S(v)}\Delta^{1/20} \le m_S^{\Delta_1}(v) = \mathrm{def}_{S_1}(v) \le \frac{1}{2}\mathrm{def}_S(v) + \frac{1}{8}\sqrt{\mathrm{def}_S(v)}\Delta^{1/20} < \frac{11}{20}\mathrm{def}_S(v)$ for every $v \in B_1^*$. So, $m_S^{\Delta_1}(B_1) \le m_S^{\Delta_1}(B_1^*) + \mathrm{def}_S(B_1 - B_1^*) < \frac{11}{20}\mathrm{def}_S(B_1^*) + 1200\Delta^{11/10}$, and similarly $m_S^{\Delta_1}(B_1) > \frac{9}{20}\mathrm{def}_S(B_1^*)$.

It follows that $m^{\Delta_1}(B_1) = m_r^{\Delta_1}(B_1) + m_S^{\Delta_1}(B_1) < \frac{11}{20}\mathrm{def}(B_1) + 1200\Delta^{11/10} < \frac{6}{10}\mathrm{def}(B_1)$ and $m^{\Delta_1}(B_1) > \frac{9}{20}\mathrm{def}(B_1) - 1200\Delta^{11/10} > \frac{4}{10}\mathrm{def}(B_1) + 4\Delta_1$.

Since $\frac{1}{2}\Delta^{12/10} - \frac{1}{4}\Delta \le \mathrm{def}(B_1) \le \Delta^{19/10} + \frac{1}{4}\Delta$ in this case, and because the marking is equalized in B_1, it follows that $m(B_1,i) < \frac{1}{\Delta_1}m^{\Delta_1}(B_1) + 2 < \frac{3}{2}\Delta^{9/10}$ and $m(B_1,i) > \frac{1}{3}\Delta^{1/5} + 4$. □

If $|B|$ is even, then $m^{\Delta_1}(B_2) - m^{\Delta_1}(B_1) = \frac{1}{2}(\mathrm{def}_S(B_2) - \mathrm{def}_S(B_1)) + \frac{1}{2}(d(S_2) - d(S_1)) + \lceil \frac{1}{2}\mathrm{def}_r(B_2)\rceil - \lceil \frac{1}{2}\mathrm{def}_r(B_2)\rceil = c_B + k$ where $k = \lceil c_S \rceil$ or $k =$

$\lfloor c_S \rfloor$. (Recall that $c_S = \frac{1}{2}(d(S_2) - d(S_1))$.) If $|B|$ is odd, then $m^{\Delta_1}(B_2) - m^{\Delta_1}(B_1) = c_B + k - \frac{1}{2}\Delta$ where $k = \lceil c_S \rceil$ or $k = \lfloor c_S \rfloor$. In both cases, $0 \leq m^{\Delta_1}(B_2) - m^{\Delta_1}(B_1) \leq \frac{1}{2}\Delta$. Since the marking is equalized in B_1 and in B_2, it follows that $|m(B_2, i) - m(B_1, i)| \leq 2$ for every $i = 1, ..., \Delta_1$. Furthermore, since $0 \leq |M_i' \cap E(B_1)| - |M_i' \cap E(B_2)| \leq 1$, it follows that $|n(B_1, i) - n(B_2, i)| \leq 4$, where $n(B_1, i)$ and $n(B_2, i)$ are the numbers of vertices in B_1 and in B_2, respectively, that are missed by and are not marked in M_i' (i.e. $m(v, i) = 0$).

Preparing the matchings and the marking for patching

In order to be able to apply our patching techniques, we require that $n(B_1, i) = n(B_2, i)$ for all $i = 1, ..., \Delta_1$. We take care of this by deleting up to 4 marks in every matching, while being careful not to delete too many marks from any one vertex:

3. For every $i = 1, ..., \Delta_1$, we delete $|z|$ marks from matching M_i', where $z = n(B_1, i) - n(B_2, i)$, as follows:

> We repeat the following $|z|$ times: if $z > 0$ (resp. $z < 0$) we pick a vertex $v \in B_2$ (resp. B_1) such that $m_r(v, i) = 1$ and v has not been picked in the previous $2\Delta^{1/10}$ iterations, or $m_u(v, i) = 1$ and neither u nor v have been chosen in the previous $2\Delta^{1/10}$ iterations. In the first case, we just set $m_r(v, i) = 0$, while in the latter we set $m_u(v, i) = 0$ and we reject (u, v).

In every iteration i, no more than $16\Delta^{1/10} < \frac{1}{3}\Delta^{6/5} \leq m(B_k, i) - 4$ marks are not available. We reject at most $4\Delta_1$ small edges and no vertex is adjacent to more than $\frac{\Delta_1}{2\Delta^{1/10}} < \frac{1}{2}\Delta^{9/10}$ rejected edges. Similarly, no vertex v lost more than $\frac{\Delta_1}{2\Delta^{1/10}}$ marks so $|m^{\Delta_1}(v) - \frac{1}{2}\text{def}(v)| < \frac{1}{2}\Delta^{9/10}$.

Claim 13 *The following properties are satisfied by the final marking (defined after step* **3.***) for $k = 1, 2$ and all $i = 1, ..., \Delta_1$:*

(i) $\frac{4}{10}\mathit{def}(B_k) < m^{\Delta_1}(B_k) < \frac{6}{10}\mathit{def}(B_k)$,

(ii) $\frac{1}{3}\Delta^{1/5} \leq m(B_k, i) < \frac{3}{2}\Delta^{9/10}$,

(iii) $\frac{1}{2}\delta \leq n(B_1^*, i) = n(B_2^*, i) < 8\delta$.

Proof: The first two conditions easily follow from our above discussion, so we only prove **(iii)**. Every $v \in B$ is unmarked in and missed by $\Delta_1 - m^{\Delta_1}(v)$ initial matchings, i.e. by at least $\frac{3}{4}\delta$ and at most $\frac{5}{4}\delta$ of the matchings $M'_1, ..., M'_{\Delta_1}$. Since these matchings are balanced in B_1 and in B_2, because the marking m is equalized in B_1 and in B_2, and because no more than $4\Delta_1$ small edges have been rejected, it follows that the number of unmarked vertices in B_k missed by M'_i is at least $(\Delta_1)^{-1}(\frac{3}{4}\delta\frac{1}{2}\Delta - 4\Delta_1) - 4 - 2 > \frac{1}{2}\delta$ and at most $(\Delta_1)^{-1}(\frac{5}{4}\delta 3\Delta) + 2 < 8\delta$. □

The patching

For $i = 1, ..., \Delta_1$, we recursively construct M_i by augmenting the vertex disjoint patches we construct in $F = B \cap (G - R_1 - R_2 - M_1 - ... - M_{i-1})$ between pairs of unmarked big vertices missed by M'_i. Note that only big edges may be rejected in the augmentations. If we fail to construct a patch between two unmarked vertices missed by some M'_i, we will show the existence of and construct a fail pair (X, Y) in (B_1, B_2).

Let $x_1, ..., x_s$ be the unmarked big vertices in B_1 missed by M'_i and let $y_1, ..., y_s$ be the unmarked big vertices missed by M'_i (where $\frac{1}{2}\delta < s < 9\delta$.) For $r = 1, ..., s$, we construct the patch P_r in between x_r and y_r as follows:

4.1 We first define **unavailable** and **usable** vertices. We call $v \in B$ **unavailable** if it belongs to any one of $P_1, ..., P_{r-1}$ or to any one of the patches constructed for one of the previous $2\lceil\Delta^{1/10}\rceil$ matchings $(M'_{i-1}, ..., M'_{i-2\lceil\Delta^{1/10}\rceil})$. We call $v \in B$ **usable** if $v = y_r$ or $(v, u) \in M'_i$ and neither v nor u is unavailable.

4.2 We recursively build the sets X^l and Y^l for $0 \leq l \leq 6\lceil\Delta^{1/20}\rceil$ as follows:

$X^0 = \{x_r\}$,
$Y^l = \{v \in B : v$ is usable and $\exists u \in X^l$ such that $(u, v) \in E(F) \cap E(B_1, B_2)\}$.
$X^l = \{v \in B : \exists u \in Y^{l-1}$ such that $(u, v) \in M'_i\}$

We observe that $Y^l \geq X^{l+1}$ for every $l = 0, ..., 6\lceil\Delta^{1/20}\rceil$ and $Y^j > X^{j+1}$ for some j if and only if $y_r \in Y^j$.

4.3 If $y_r \in Y^j$ for some $0 \leq j \leq 6\lceil\Delta^{1/20}\rceil$, we construct the patch defined by the sequence of vertices $x_r = x^0, y^0, x^1, y^1, ..., y^{j-1}, x^j, y^j = y_r$ where $x^l \in X^l$, $y^l \in Y^l$, $(x^l, y^l) \in E(F) \cap E(B_1, B_2)$ and $(y^l, x^{l+1}) \in M'_i$.

We observe that each patch contains the same number of edges from $E(B_1)$ and from $E(B_2)$ and that in every augmentation an equal number of edges in $E(B_1)$ and in $E(B_2)$ are rejected. It follows that $|E(R_1) \cap E(B_1)| = |E(R_2) \cap E(B_2|$. Furthermore, we reject at most $\frac{\Delta_1}{2\Delta^{1/10}} < \frac{1}{2}\Delta^{9/10}$ edges incident to any particular vertex. Finally the total number of edges in B_1 or in B_2 we reject is less than $\Delta_1 10\delta 6\lceil\Delta^{1/20}\rceil < \frac{1}{12}\Delta^{19/10}$.

4.4 If there is no Y^j containing y_r, then we pick the smallest $j \geq 1$ such that $|Y^j| \leq |X^j| + \frac{1}{2}\Delta^{19/20}$. We will show in claim 14 below that $j < 6\lceil\Delta^{1/20}\rceil - 2$. Let F_1 and F_2 be the vertices in B_1 and B_2, respectively, that are not usable, and let E_1 and E_2 be subsets of B_1 and B_2, respectively, missed by the big edges of M_i'. We set $Y = Y^j \cup F_2 \cup E_2$ if $Y^j \subset B_2$ or $Y = Y^j \cup F_1 \cup E_2$ if $Y^j \subset B_1$ and $X = \{v \in X^j : d_Y^F(v) > \frac{1}{2}\Delta - \Delta^{19/20}\}$.

Claim 14 *(X, Y) is a fail pair in (B_1, B_2).*

We note that $d_B^F(v) > \frac{1}{2}\text{def}(v) - \Delta^{9/10}$ for all $v \in B$ since 4.3 is satisfied, implying $d_{B_2}^F(v) > \frac{1}{2}\text{ def}(v) - 2\Delta^{9/10}$ for all $v \in B_1$ and $d_{B_1}^F(v) > \frac{1}{2}\text{def}(v) - 2\Delta^{9/10}$ for all $v \in B_2$.

Proof: To simplify notation, let $x = x_r \in B_1$ and $y = y_r \in B_2$.

Since a patch contains at most $6\lceil\Delta^{1/20}\rceil$ vertices in B_1, and since there are at most 8δ patches per matching, it follows that

$$|F_1| < 2\lfloor\Delta^{1/10}\rfloor 8\delta 6\lceil\Delta^{1/20}\rceil < \frac{1}{8}\Delta^{19/20}$$

A symmetrical argument gives $|F_2| < \frac{1}{8}\Delta^{19/20}$.

It follows that $|X^1| = |Y^0| \geq d_{B_2}^F(v) - |F_2| > \frac{1}{4}\Delta - \frac{1}{4}\Delta^{19/20}$. If we assume that $|Y^l| > |X^l| + \frac{1}{2}\Delta^{19/20}$ for all $1 \leq l \leq \lceil 6\Delta^{1/20}\rceil - 2$, then $|X^{6\lceil\Delta^{1/20}\rceil-2}| > 3\Delta \geq |B_1|$, a contradiction.

So we must have $|Y^j| \leq |X^j| + \frac{1}{2}\Delta^{19/20}$ for some j between 1 and $\lceil 6\Delta^{1/20}\rceil - 2$, and we pick the minimum j satisfying this property. Then the pair (X, Y), as constructed in step **4.4** forms a fail pair in (B_1, B_2) if the following 3 conditions are satisfied:

$|Y| \leq |X| + \Delta^{19/20}$.

Proof: We assume $X \subset B_1$ and $Y \subset B_2$; a symmetric argument follows when $X \subset B_2$ and $Y \subset B_1$. We note that $|Y| = |Y^j| + |E_2| + |F_2| \leq |X^j| + \frac{1}{2}\Delta^{19/20} + 8\delta + \frac{3}{2}\Delta^{9/10} + \frac{1}{8}\Delta^{19/20} < |X^j| + \frac{3}{4}\Delta^{19/20}$. If $v \in X^j - X$ then $d^F_{B_2}(v) < \frac{1}{2}\Delta - \Delta^{19/20}$ and $d_B(v) < \Delta - \Delta^{19/20}$, implying $v \in B^- \cap B_1$. Since $|B^- \cap B_1| \leq \frac{1}{4}\Delta^{19/20}$, **i** follows.

For all $v \in X$: $d_Y(v) > \frac{1}{2}\Delta - \Delta^{19/20}$.

Proof: This follows by the definition of X.

If $X \subset B_1$ then $|B_1| - |X| > \frac{1}{2}\Delta - \Delta^{19/20}$, or if $X \subset B_2$ then $|B_2| - |X| > \frac{1}{2}\Delta - \Delta^{19/20}$.

Proof: If $X \subset B_1$ then y must belong to $B_2 - Y$ and $d^F_X(y) = 0$. Since $d^F_{B_1}(y) > \frac{1}{4}\Delta - 2\Delta^{9/10}$, it follows that $|B_1| - |X| > \frac{1}{4}\Delta - \Delta^{19/20}$.

If $X \subset B_2$, then $d^F_{X^{j+1}}(y) = 0$ and $d^F_{B_1}(y) > \frac{1}{4}\Delta - 2\Delta^{9/10}$, so that $|B_1 - Y^j| = |B_1 - X^{j+1}| > \frac{1}{4}\Delta - \Delta^{9/10}$. So, there must exist a vertex $v \in B_1 - X^{j+1}$ with $(v, u) \in M'_i$ such that v and u are usable. Since $v \in B_1 - X^{j+1}$ however, $d^F_X(v) = 0$. But $d^F_{B_1}(v) > \frac{1}{4}\Delta - 2\Delta^{9/10}$ and it follows that $|B_1 - X| > \frac{1}{4}\Delta - \Delta^{19/20}$.

□

4.2.2 The reject coloring pass

After the first pass has been completed and the matchings $M_1, ..., M_{\Delta_1}$ have been deleted, we obtain the reduction $F = G - \cup_{i=1}^{\Delta_1} M_i$ of G. In the reject coloring pass, we construct the disjoint matchings $M_{\Delta_1+1}, ..., M_{\Delta_1+\Delta_2}$ such that $\cup_{i=1}^{\Delta_2} M_{\Delta_1+i}$ contains $E(R_1) \cup E(R_2)$ and $H = F - \cup_{i=1}^{\Delta_2} M_{\Delta_1+i}$ has degree $\Delta - \Delta_1 - \Delta_2$, so that H is a bipartite reduction of G.

The initial matchings

Let k be the largest integer less than or equal to $\frac{1}{6}\Delta^{19/20}$ such that $\Delta_2 - k$ is even (recall that $\Delta_2 = \lceil \frac{1}{2}\Delta^{19/20} \rceil$, so that $\Delta_2 - k \geq \frac{1}{3}\Delta^{19/20}$). We first color the small edges of R_1 and R_2 with k colors:

1.1 We color $E(R_1) \cap E(B_1, S_1)$ and $E(R_2) \cap E(B_2, S_2)$ with k colors using Fournier's algorithm and we obtain disjoint matchings $M'_{\Delta_1+1}, ..., M'_{\Delta_1+k}$ such that

$$\cup_{i=1}^{k} M'_{\Delta_1+i} = (E(R_1) \cap E(B_1, S_1)) \cup (E(R_2) \cap E(B_2, S_2))$$

We color the big reject edges with the remaining $\Delta_2 - k$ colors:

1.2 If $|B|$ is even, we color $E(R_1) \cap E(B_1)$ and $E(R_2) \cap E(B_2)$ with $\Delta_2 - k$ colors and we use the procedure from 3.2.2 to obtain disjoint matchings $M'_{\Delta_1+k+1}, ..., M'_{\Delta_1+\Delta_2}$, balanced in B_1 and in B_2, such that

$$\cup_{i=k+1}^{\Delta_2} M'_{\Delta_1+i} = (E(R_1) \cap E(B_1)) \cup (E(R_2) \cap E(B_2))$$

Since $|E(R_1) \cap E(B_1)| = |E(R_2) \cap E(B_2)| < \frac{1}{12}\Delta^{19/20}$ it follows that $|M'_{\Delta_1+i} \cap E(B_1)| = |M'_{\Delta_1+i} \cap E(B_2)| < \frac{1}{4}\Delta^{19/20}$ for every $i = k+1, ..., \Delta_2$.

1.3 If $|B|$ is odd, we color $E(R_1) \cap E(B_1)$ and $E(R_2) \cap E(B_2)$ with $l = \frac{1}{2}(\Delta_2 - k)$ colors and we use the procedure from 3.2.2 to obtain temporary disjoint matchings $M''_1, ..., M''_l$ balanced in B_1 and in B_2 such that

$$\cup_{i=1}^{l} M''_i = (E(R_1) \cap E(B_1)) \cup (E(R_2) \cap E(B_2))$$

Note that $|M''_i \cap E(B_1)| = |M''_i \cap E(B_2)| < \frac{1}{2}\Delta^{19/20}$ for every $i = 1, ..., l$.

The marking

We define a marking and, in the process, modify the matchings M'_{Δ_1+1}, ..., $M'_{\Delta_1+\Delta_2}$ so that $n(B_1, \Delta_1 + i) = n(B_2, \Delta_1 + i)$ for all $i = 1, ..., \Delta_2$. We define the marking in the first k matchings, for $i = 1, ..., k$, as follows:

2.1.1 For every $i = 1, ..., k$ and for every edge $(u, v) \in M'_{\Delta_1+i}$ (where $u \in S$ and $v \in B$) we set $m_u(v, \Delta_1 + i) = 1$.

2.1.2 We balance the marking in B_1 and B_2 over the matchings $M'_{\Delta_1+1}, ...,$ M'_{Δ_1+k} using the balancing procedure from 3.2.3.

Since $||E(R_1) \cap E(B_1, S_1)| + |E(R_2) \cap E(B_2, S_2)|| \leq 4\Delta_1$, it follows that $|M'_{\Delta_1+i}| < 25\Delta^{1/20} < \Delta^{\frac{1}{10}}$ for all $i = 1, ..., k$.

After **2.1.2**, the difference between the number of marks in B_1 and in B_2 can be as high as $\Delta^{1/10}$. To obtain $n(B_1, \Delta_1+i) = n(B_2, \Delta_1+i)$ for all $i = 1, ..., k$, we must put additional marks to every matching $M'_{\Delta_1+1}, ..., M'_{\Delta_1+k}$. To describe this additional marking, we find it useful to denote $\sum_{j=1}^{i} m(v,i)$ by $m_u^i(v)$.

2.1.3 We iteratively add additional marks to every M'_{Δ_1+i} for $i = 1, ..., k$ as follows:

If there are z more unmarked vertices in B_1 than in B_2 we pick z different vertices $v_1, ..., v_z$ in B_1 with $\mathrm{def}_{u_1}(v_1) > m_{u_1}^{i-1}(v_1), ..., \mathrm{def}_{u_z}(v_z) > m_{u_z}^{i-1}(v_z)$ where $u_j = r$ or $u_j \in S_2$ for every j. We insist that $u_l = u_k$ if and only if $u_l = u_k = r$. We then set $m_{u_1}(v_1, \Delta_1 + i) = 1$, ..., $m_{u_z}(v_z, \Delta_1 + i) = 1$. If B_2 has more unmarked vertices than B_1, we use the obvious symmetric procedure.

The remaining deficiency after the first coloring pass is greater than $\frac{1}{6}\Delta^{12/10}$. The total that could be used in **2.1.1 - 2.1.3** is less than $2k\Delta^{1/10} < \frac{1}{2}\Delta^{21/20}$. So, at any iteration of **2.1.3**, there is at least $\frac{1}{12}\Delta^{12/10}$ available deficiency, implying that we can choose $v_1, ..., v_z, u_1, ..., u_z$ greedily.

We now look at the remaining $\Delta_2 - k$ matchings. If $|B|$ is even, we actually put no marks in these matchings. If $|B|$ is odd, however, we must put a mark in B_1 in each matching.

2.2 If $|B|$ is odd, for $i = 1, ..., l$, we pick two different vertices v and v' in B_1 with $\mathrm{def}_{u_1}(v_1) > m_{u_1}^{i-1}(v_1)$ and $\mathrm{def}_{u_2}(v_2) > m_{u_2}^{i-1}(v_2)$ such that $(v_1, v_2) \notin M''_i$. (We easily can do this by the same argument as above.) We then split M''_i into $M'_{\Delta_1+k+2i-1}$ and M'_{Δ_1+k+2i} such that $M'_{\Delta_1+k+2i-1}$ misses v_1, M'_{Δ_1+k+2i} misses v_2, $|M'_{\Delta_1+k+2i-1} \cap E(B_1)| = |M'_{\Delta_1+k+2i-1} \cap E(B_2)| \leq \frac{1}{4}\Delta^{9/10}$ and $|M'_{\Delta_1+k+2i} \cap E(B_1)| = |M'_{\Delta_1+k+2i} \cap E(B_2)| \leq \frac{1}{4}\Delta^{9/10}$. Finally, we set $m_{u_1}(v_1, \Delta_1 + k + 2i - 1) = 1$ and $m_{u_2}(v_2, \Delta_1 + k + 2i) = 1$.

When done $m(B_1, \Delta_1 + i), m(B_2, \Delta_1 + i) < \Delta^{1/10}$ and $n(B_1, \Delta_1 + i) = n(B_2, \Delta_1 + i)$ for $i = 1, ..., \Delta_2$.

The patching

For $i = 1, ..., \Delta_1$, we recursively construct M_{Δ_1+i} by augmenting M'_{Δ_1+i} in $H = F - M_{\Delta_1+1} - ... - M_{\Delta_1+i-1}$ so that all unmarked big vertices missed by M'_{Δ_1+i} are hit by a big edge of M_{Δ_1+i}:

3. Let U_1 and U_2 be the sets of unmarked big vertices missed by M'_{Δ_1+i}. We attempt to construct a perfect matching M^* in the graph induced by the vertex partition (U_1, U_2) in H. If successful, we obtain M_{Δ_1+i} by adding M^* to M'_{Δ_1+i}.

If not successful, we find the sets $X' \subset U_1$ and $Y' = N^H_{U_2}(X')$ such that $|X'| > |Y'|$. Let F_1 and F_2 be the marked vertices in B_1 and in B_2, respectively, and let E_1 and E_2 be the vertices in B_1 and in B_2, respectively, that are hit by a big edge of M'_i. We set $Y = Y' \cup F_2 \cup E_2$ and $X = \{v \in X' : d_Y(v) > \frac{1}{2}\Delta - \Delta^{19/20}\}$.

Claim 15 *(X, Y) forms a fail pair in (B_1, B_2).*

Proof: The following three properties must hold:

$|Y| < |X| + \Delta^{19/20}$.
Proof: $|Y| = |Y'| + |F_2| + |E_2| \leq |X'| + \Delta^{1/10} + \frac{1}{2}\Delta^{19/20} < |X| + \Delta^{19/20}$.

$d_Y(v) > \frac{1}{2}\Delta - \Delta^{19/20}$ for all $v \in X$.
Proof: By definition of X.

$|B_1 - X| > \frac{1}{4}\Delta - \Delta^{9/10}$.
Proof: Since X'' is non-empty, there exists $v \in X''$. Since $X'' \subset U_2 \subset B_2$, v is a big vertex and $d^H_X(v) = 0$. Then $|B_1 - X| \geq d^H_{B_1}(v) > d^F_{B_1}(v) - \Delta_2 > \frac{1}{4}\Delta - \Delta^{19/20}$.

□

Chapter 5

The small deficiency case

If the Vizing graph $G = (B \cup S, E)$ has deficiency less than $\Delta^{12/10}$, we find it convenient to identify most of the small vertices, while keeping multiple edges, so that the degree of all but at most 3 small vertices is more than $\frac{1}{2}\Delta$ (and of course at most Δ). Then, with some care, we treat these new "high degree small vertices" as if they were big.

We start this chapter with a description of the identification procedure and the statement of the modified main theorem, a theorem about Δ edge coloring large degree graphs with identified small vertices that is equivalent to the main theorem. In section 5.2, we discuss the modifications we must make to the definition of a split partition in order to accomodate the multiple edges created by the identifications. We also describe a randomized procedure that with positive probability returns a "modified" split partition. Finally, we present the first of our two edge-coloring algorithms for graphs of small deficiency in section 5.3. We leave the last, more technical algorithm for graphs of smallest deficiency to chapter 6.

5.1 The identification procedure

We now identify the vertices in S of a Vizing graph $G = (B \cup S, E)$ to obtain the graph $G^* = (B^* \cup S^*, E^*)$ where B^* is the set of vertices of degree greater than $\frac{1}{2}\Delta$ including all vertices in B and all identifications of degree greater than $\frac{1}{2}\Delta$ and S^* is the set of vertices of degree at most $\frac{1}{2}\Delta$. We use a recursive procedure to construct G^*. In order to describe this recursive identification procedure we extend the notion of deficiency to every vertex v in B^*: let $\mathrm{def}(v) = \Delta - d_{B^*}(v)$, $\mathrm{def}_r(v) = \Delta - d(v)$ and $\mathrm{def}_{S^*}(v) = \mathrm{def}(v) - \mathrm{def}_r(v) = d_{S^*}(v)$.

We describe now the identification procedure. We set, initially, $B^* = B$ and $S^* = S$. We identify the vertices in S^* recursively a follows:

> If $|S^*| \geq 2$ and $\mathrm{def}(B^*) \geq 2(\mathrm{def}_{s'}(B^*) + \mathrm{def}_{s''}(B^*))$, where s' and s'' are the two smallest degree vertices in S^*, we identify s' and s'' to obtain s^* while keeping any multiple edges; otherwise we stop. We put s^* into S^* if its degree is at most $\frac{1}{2}\Delta$ or into B^* if its degree is more than $\frac{1}{2}\Delta$.

We call the resulting multigraph $G^* = (B^* \cup S^*, E^*)$ a **multi-Vizing reduction** of the Vizing graph $G = (B \cup S, E)$. We call vertices in B^* **big** and vertices in S^* **small**. Note that B^* includes all (non-identified) vertices of B. We will also call big all edges in $E(B^*)$ and small all edges in $E(B^*, S^*)$.

Before discussing how the multi-Vizing reduction G^* can help us prove the main theorem, we need to understand the properties that G^* satisfies:

Claim 16 *A multi-Vizing reduction $G^* = (B^* \cup S^*, E^*)$ of a Vizing graph $G = (B \cup S, E)$ of deficiency less than $\Delta^{12/10}$ satisfies:*

a. *def$(B^*) < \Delta^{12/10}$, $|B^* - B| < 2\Delta^{1/5}$ and $d_{B^*-B}(v) < \Delta^{3/5}$ for every $v \in B$,*

b. *The graph induced by vertices of B in G^* is equal to the same graph in G. $(B^* \cup S^*) - B$ is an independent set with $|S^*| \leq 3$ and if def$(B^*) \geq 2\Delta$ then $|S^*| \leq 1$,*

c. *The multiplicity of an edge in $E((B^* \cup S^*) - B, B)$ is less than $\sqrt{\Delta}$; if $\mu(u,v) \geq 2$ for some vertices $u \in B^* - B$ and $v \in B$ such that def(u), def$(v) > \frac{1}{4}\Delta - \Delta^{9/10}$ then $\mu(u,v) \leq 3$ and def$(v) < \frac{3}{8}\Delta + \Delta^{9/10}$,*

d. *$G^* = (B^* \cup S^*, E^*)$ is weakly Vizing, i.e. $d_{B^*}(v) > \frac{1}{2}\Delta$ for every $v \in B^*$, $d_{B^*}(v) \leq \frac{1}{2}\Delta$ for every $v \in S^*$, and $d_{B^*}(v) + d_{B^*-v}(u) \geq \Delta$ for all $u, v \in B^*$.*

Note that we extend the definition of a weakly Vizing graph from simple graphs to multigraphs.

Proof: At any step of the identification procedure, two vertices s' and s'' in S^* are identified to obtain s^*. If $d(s^*) > \frac{1}{2}\Delta$ then s^* is put into B^* which decreases the fake deficiency of B^* by $d(s^*) > \frac{1}{2}\Delta$, but which also increases the real deficiency of B^* by $\Delta - d(s^*) < \frac{1}{2}\Delta$. If $d(s^*) \leq \frac{1}{2}\Delta$ then s^* is put into S^* and the deficiency of B^* stays the same. So, the deficiency of B^* that we obtain through the identification procedure cannot be greater than def(B), and it follows that def$(B^*) \leq \Delta^{12/10}$. We note that the number of edges between $B^* \cup S^* - B$ and B is also at most $\text{def}_{S^*}(B) < \Delta^{12/10}$. Since every vertex in $B^* - B$ has degree greater than $\frac{1}{2}\Delta$ we obtain $|B^* - B| < 2\Delta^{1/5}$. The last point we need to prove property **a** is that $d^{G^*}_{B^*-B}(v) < \Delta^{3/5}$ for every $v \in B$. We first note that $d^G_B(u) > d^G_S(v)$ for every $u \in S$ adjacent to any $v \in B$ (since $G = (B \cup S, E)$ is Vizing). It follows that no vertex in B can have $\Delta^{3/5}$ edges to small vertices of G (otherwise, $\text{def}(B) > (d^G_S(v))^2 \geq \Delta^{12/10}$). This implies that $d^{G^*}_{B^*-B}(v) < \Delta^{3/5}$.

Property **b.** holds for G^* by construction.

If an edge $(u, v) \in E(B^* \cup S^* - B, B)$ has multiplicity $m \geq \sqrt{\Delta}$ then u has been obtained partly by identifying m small neighbors of v in G; by the earlier remark, however, each small neighbor of v has degree more than $d_S(v) \geq m$ in G, which would imply that u has degree more than $m^2 \geq \Delta$ in G^*. If $\mu(u, v) \geq 2$ for some vertices $u \in B^* - B$ and $v \in B$ such that $\text{def}(u), \text{def}(v) > \frac{1}{4}\Delta - \Delta^{9/10}$ then the deficiency of v in G is at least $\frac{1}{4}\Delta - \Delta^{9/10} - \Delta^{3/5} > \frac{1}{4}\Delta - 2\Delta^{9/10}$. Then, every small neighbor (i.e in S) of v in G has degree greater than $\frac{1}{4}\Delta - 2\Delta^{9/10}$. Since the degree of u in G^* is less than $\frac{3}{4}\Delta + \Delta^{9/10}$, no more than three small neighbors of v in G are identified to form u, implying $\mu(u, v) \leq 3$. Furthermore, if the deficiency of v in G^* is greater than $\frac{3}{8}\Delta + \Delta^{9/10}$ then the deficency of v in G is greater than $\frac{3}{8}\Delta + \frac{1}{2}\Delta^{9/10}$ and every small neighbor of v in G has degree greater than $\frac{3}{8}\Delta + \frac{1}{2}\Delta^{9/10}$, implying that the degree of u in G^* is at greater than $\frac{3}{4} + \Delta^{9/10}$ contradicting our assumption.

Finally, we show that property **d** holds, i.e we show that $d_{B^*}(v) + d_{B^*-v}(u) \geq \Delta$ for any two vertices u and v in B^*. This is true if vertices u and v are both in B (since the graph induced by B is weakly Vizing itself) or both in $B^* - B$ (since $B^* - B$ forms an independent set). If $v \in B$ and $u \in B^* - B$ then $d_{B^*}(v) + d_{B^*-v}(u) \geq \Delta$ holds if they are not adjacent. If they are adjacent, let $u' \in S$ be a neighbor of v in G that was identified with some other small vertices to obtain u; then $d^{G^*}_{B^*-v}(u) + d^{G^*}_{B^*}(v) \geq d^G_{B-v}(u') + |\{x \in B : (x, v) \in E \text{ and } d^G(x) = \Delta\}| \geq \Delta$ follows since G is Vizing. $\square$

We will often just speak of a multi-Vizing reduction $G^* = (B^* \cup S^*, E^*)$ and assume that it is obtained from a Vizing graph $G = (B \cup S, E)$ of deficiency less than $\Delta^{12/10}$.

We can discuss now how G^* can help us prove the main theorem. We first note the obvious fact that a Δ edge coloring of G^* is easily extendable to G. Furthermore, if a subgraph H of B is forbidden in G^*, then H is also forbidden in G; recall that a subgraph H of B is **forbidden** if its minimum degree $\delta(H) \geq \Delta - \Delta^{79/80}$ and either:

(i) H is bipartite, or

(ii) $|B - H| > \frac{1}{2}\Delta - \Delta^{79/80}$

Let F be an overfull subgraph of G^* of maximum degree Δ (recall that F is overfull if $E(F) > \frac{1}{2}\Delta(|V(F)| - 1)$). We extend the notion of a trivial overfull subgraph, originally defined in the case of Vizing graphs, to multi-Vizing reductions as follows: F is trivial if it has degree Δ and $F = B^* - v$ for some $v \in B^*$, $F = B^*$ or $F = B^* + u$ for some $u \in S^*$. We also extend the trivial lemma (lemma 17) as follows:

Lemma 20 (multi-trivial) *Let $G^* = (B^* \cup S^*, E^*)$ be a multi-Vizing reduction of the Vizing graph $G = (B \cup S, E)$ of maximum degree Δ containing an overfull subgraph of degree Δ. Then one of the following must hold:*

(i) *G^* contains a trivial overfull subgraph, or*

(ii) *G^* contains a forbidden subgraph H in B.*

In addition we will show below that:

Lemma 21 *Let $G^* = (B^* \cup S^*, E^*)$ be a multi-Vizing reduction of the Vizing graph $G = (B \cup S, E)$ of maximum degree Δ. If G^* contains a trivial overfull subgraph F then F is a trivial overfull subgraph of G too.*

The above discussion and the two lemmas 20 and 21 show that in order to prove the main theorem for Vizing graphs of deficiency less than $\Delta^{12/10}$, it is sufficient to prove:

Theorem 22 *There exists Δ_0 such that for all Vizing graphs $G = (B \cup S, E)$ of maximum degree $\Delta > \Delta_0$ satisfying $|B \cup S| \leq 6\Delta$ and def$(B) \leq \Delta^{12/10}$, one of the following holds for the multi-Vizing reduction $G^* = (B^* \cup S^*, E^*)$ of G:*

(i) *G^* contains a trivial overfull subgraph,*

(ii) *B contains a forbidden subgraph H,*

(iii) *G^* is Δ edge colorable.*

Furthermore, there is a procedure which runs in $O(2^{|V|})$ time that will output either a Δ edge coloring of G^*, a trivial overfull subgraph of G^* or a forbidden subgraph of B.

We prove this theorem by designing, and proving the correctness of, two algorithms that attempt to color with Δ colors the edges of a graph of small deficiency. We apply the first algorithm (described in section 5.3 of this chapter) to graphs with $2\Delta < \text{def}(B^*) < \Delta^{12/10}$. The second, more technical algorithm deals with graphs with $\text{def}(B^*) < \frac{5}{2}\Delta$ and is dicussed in chapter 6. We will show that both algorithms fail only if G^* contains a trivial overfull subgraph or B contains a forbidden subgraph. We now prove lemma 20:

Proof: Let F be an overfull subgraph of G^* of maximum degree Δ, and let us slightly abuse the notation by denoting by F the set of vertices of the graph F.

We first remark that the set $R = \{v \in F : d_F(v) < \Delta - \sqrt{\Delta}\}$ is smaller than $\sqrt{\Delta}$. Thus the graph H induced by $(F-R) \cap B$ has minimum degree greater than $\Delta - 2\sqrt{\Delta} - \Delta^{3/5}$ (since $d_{B^*-B}(v) < \Delta^{3/5}$ for every $v \in B^*$ by claim 16). If H is not a forbidden subgraph of B then $|B - H| \leq \frac{1}{2}\Delta - \Delta^{79/80}$.

Consider the set $C = \{v \in B - F : d_F(v) > \sqrt{\Delta}\}$. Clearly, $|C| < \sqrt{\Delta}$. Note that $d_B(v) > \frac{1}{2}\Delta$ for any v in B and $d_F(v) \leq \sqrt{\Delta}$ for any $v \in B - F - C$. So, if $B - F - C$ is not empty then $|B - F| > \frac{1}{2}\Delta - \sqrt{\Delta}$ which in turn gives $|B - H| > \frac{1}{2}\Delta - \Delta^{79/80}$ contradicting our assumption that H is not forbidden (note that $|F - H| \leq |R| + |B^* - B| < 2\sqrt{\Delta}$. Clearly then, $B - F = C$ and $|B - F| \leq \sqrt{\Delta}$. Note that if $|B - F| > 2$ then $|E(F, B-F)| > \frac{3}{2}|\Delta - 3\sqrt{\Delta} > \Delta$ which contradicts the fact that F is overfull. We thus obtain $|B - F| \leq 2$.

If $B - F = \{u, v\}$, then $|E(F + u + v)| = |E(F)| + d_B(u) + d_{B-u}(v) > \frac{1}{2}\Delta(|V(F)| - 1) + \Delta = \frac{1}{2}\Delta(|F + u + v| - 1)$ and $F + u + v$ is an overfull subgraph of G as well. So, in any case, there exists an overfull subgraph F' of maximum degree Δ such that $|B - F'| \leq 1$. A similar argument also show that $|B^* - F| \leq 1$.

Suppose now that $|F' \cap S^*| \geq 2$, and let u and v be two vertices of $F' \cap S^*$. Then $|E(F')| = |E(F' - u - v)| + d_F(u) + d_F(v) \leq \frac{1}{2}\Delta(|F'| - 2) - (d_F(u) + d_F(v)) + (d_F(u) + d_F(v)) = \frac{1}{2}\Delta(|F'| - 2)$ contradicting the fact that F' is overfull. So, $|F' \cap S^*| \leq 1$.

Finally, we show that if $|B - F'| = 1$ and $|S \cap F'| = 1$, then B itself must be overfull. This would prove our claim that G^* must contain a trivial overfull subgraph. Suppose that $F' = B^* - v + u$ for some $v \in B^*$ and $u \in S^*$. Then $|E(F')| = |E(F' - u)| + d_F(u) \leq |E(B^*)| - d_{B^*}(v) + d_{B^*}(u) \leq |E(B^*)|$ since $d_{B^*}(v) \geq d_{B^*}(u)$. Since $|E(F')| > \frac{1}{2}\Delta(|F'| - 1) = \frac{1}{2}\Delta(|B^*| - 1)$, B^* must be overfull.

□

Finally we prove lemma 21:

Proof: It is sufficient to prove that each identification satisfies this property. Let G^* be obtained from G by identifying s' and s'' in S into s^* in S^*. Let F be the overfull graph induced by B^*, $B^* - v$ for some $v \in B^*$ or $B^* + s$ for some $u \in S^*$. If $s^* \notin V(F)$, it is easy to see that F is also a trivial overfull subgraph of G. We now show that s^* cannot belong to $V(F)$. This will prove our lemma.

Assume $s^* \in V(F)$. If $V(F) = B^*$ then since F is overfull in G^* we have that $\mathrm{def}(B^*) < \Delta$. If $V(F) = B^* - b$ then since F is overfull in G^* we have $\mathrm{def}(B^*) < 2\mathrm{def}(v) < \Delta$. Finally if $V(F) = B^* + u$ then since F is overfull in G^* we have $\mathrm{def}(B^*) < 2\mathrm{def}_u(B^*) \leq \Delta$. However, $\mathrm{def}(B^*) < \Delta$ implies that in G

$$\mathrm{def}(B^*) - \mathrm{def}_{s'}(B^*) - \mathrm{def}_{s''}(B^*) + (\Delta - \mathrm{def}_{s'}(B^*) - \mathrm{def}_{s''}(B^*)) < \Delta$$

(where the deficiencies are taken in G!) which contradicts our identification assumption that $\mathrm{def}(B^*) \geq 2(\mathrm{def}_{s'}(B^*) - d_{s''}(B^*))$. □

5.2 A modified split partition

In 2.5.1 we defined and constructed a split partition $(B_1 \cup S_1, B_2 \cup S_2)$ of a simple Vizing graph $G = (B \cup S, E)$. We define, similarly, a modified split partition $(B_1^* \cup S_1^*, B_2^* \cup S_2^*)$ of the vertices of a multi-Vizing reduction $G^* = (B^* \cup S^*, E^*)$. In a partition of the vertices of G^*, we require the degree of any vertex, and not just its neighborhood, to split about evenly between the two sides of the bipartition. While this is a non-issue in simple Vizing graphs, it is something we must worry about in multi-Vizing reductions with multiple edges.

Recall that S^* may have up to 3 vertices: let us add additional vertices so $S^* = \{s_1, s_2, s_3\}$ where $d(s_1) \geq d(s_2) \geq d(s_3) \geq 0$. Let $b_1, b_2, ..., b_{10}$ be the ten largest deficiency vertices in B^* such that $\text{def}(b_1) \geq \text{def}(b_2) \geq ... \geq \text{def}(b_{10})$. We set Δ_1 to be the smallest even (odd) integer greater than $\frac{1}{2}\Delta + \Delta^{3/4} \ln n$ if $|B^*|$ is even (odd), and $\delta = \Delta_1 - \frac{1}{2}\Delta$. A partition $(B_1^* \cup S_1^*, B_2^* \cup S_2^*)$ of $B^* \cup S^*$ is called a **modified split partition** if the following are satisfied:

(a) $B_1^* \cup B_2^* = B^*$ and $0 \leq |B_1^*| - |B_2^*| \leq 1$, $S_2^* = \{s_1, s_3\}$ and $S_1^* = \{s_2\}$.

(b) For all v in B and for all $X, Y \subset B$ of size less than $20 \log \Delta$ the following sets split within $\frac{1}{4}\Delta^{11/20}$:

$$N_B(v), N_B(X), N_B(v) \cap N_B(X), \{w \in N_B(X) : d_{N_B(Y)}(w) > \Delta - 7\Delta^{39/40}\}$$

(c) Let $B_1 = B \cap B_1^*$ and $B_2 = B \cap B_2^*$. Then $|d_{B_1}(v) - d_{B_2}(v)| \leq \frac{1}{4}\Delta^{3/4} \log \Delta$ for all v in $(B^* - B) \cup S^*$.

(d) If $|B^*|$ is even, $b_i \in B_2^*$ if i is odd, $b_i \in B_1^*$ if i is even and $\text{def}(B_1^* - b_2 - b_4 - b_6 - b_8 - b_{10}) \leq \text{def}(B_2^* - b_1 - b_3 - b_5 - b_7 - b_9)$; if $|B^*|$ is odd then $b_i \in B_2^*$ if i is even, $b_i \in B_1^*$ if i is odd and $\text{def}(B_1^* - b_1 - b_3 - b_5 - b_7 - b_9) \leq \text{def}(B_2^* - b_2 - b_4 - b_6 - b_8 - b_{10})$;

Let $c_B = |E(B_1)| - |E(B_2)|$. If $|B^*|$ even, property **(a)** implies that $c_B = \frac{1}{2}(\text{def}(B_2^*) - \text{def}(B_1^*))$. Property **(d)** then gives $0 \leq c_B \leq \frac{1}{2}\text{def}(b) < \frac{1}{4}\Delta$. If $|B^*|$ is odd, property **(a)** implies that $c_B = \frac{1}{2}(\Delta - (\text{def}(B_1^*) - \text{def}(B_2^*)))$. Again, it follows from property **(d)** that $\frac{1}{4}\Delta < \frac{1}{2}(\Delta - \text{def}(b)) \leq c_B \leq \frac{1}{2}\Delta$. In order to prove that a modified split partition exists, we show that the following procedure constructs with positive probability a modified split partition $(B_1^* \cup S_1^*, B_2^* \cup S_2^*)$ of $B^* \cup S^*$:

> We order the vertices in B^* by non-decreasing deficiency. For each successive ordered pair of vertices we switch the order of the pair with probability $1/2$ and put the first vertex in the set B_1^* and the second in the set B_2^*. If $|B^*|$ is even, after all the vertices but the last 5 pairs have been assigned to B_1^* or B_2^*, we rename B_1^* and B_2^* so that $\mathrm{def}(B_1^*) \leq \mathrm{def}(B_2^*)$ and we add b_1, b_3, b_5, b_7 and b_9 to B_2^* and b_2, b_4, b_6, b_8 and b_{10} to B_1^*. If $|B^*|$ is odd, after all the vertices but b_1 through b_9 have been assigned to B_1^* or B_2^*, we rename B_1^* and B_2^* so that $\mathrm{def}(B_1^*) \leq \mathrm{def}(B_2^*)$ and we add b_1, b_3, b_5, b_7 and b_9 to B_1^* and b_2, b_4, b_6 and b_8 to B_1^*. When done, we assign the vertices in S^*, if any, as follows: we put s_1 and s_3 into S_2^* and s_2 into S_1^*.

It is easy to see that the resulting partition satisfies conditions **(a)** and **(d)** of a modified split partition. Furhermore, our argument in claim 4 of section 2.5.1 shows that condition **(b)** holds with probability at least $\frac{1}{2}$. In the claim below, we prove that property **(c)** doesn't hold with probability at most $\frac{1}{4}$. So, the partitioning procedure returns a modified split partition with probability at least $\frac{1}{4}$.

Claim 17 $|d_{B_1}(v) - d_{B_2}(v)| \leq \frac{1}{4}\Delta^{3/4}\ln\Delta$ *for all* v *in* $(B^* - B) \cup S^*$.

Proof: It is enough to show that the probability is less then $\frac{1}{4n}$ for any particular $v \in (B^* - B) \cup S^*$. Let $N = \{v_1, ..., v_k\}$ be the neighbors of some $v \in (B^* - B) \cup S^*$, and let $H = \{x_1, ..., x_k\}$ be the set of corresponding multiplicities (i.e. $\mu(v, v_i) = x_i$ for all $i = 1, ..., k$.) We assume that no two vertices in N are paired in the partition step (if such pairs exist, then we can replace the pair, for this analysis, with one vertex whose corresponding multiplicity is equal to the difference between the multiplicities corresponding to the two original vertices). We also assume that N does not contain $b_1, ..., b_{10}$ as their placement in the partitioning procedure is not random.

We observe that $1 \leq x_i < \sqrt{\Delta}$ for all $i \leq 1, ..., k$ and that $x_1 + x_2 + ... + x_k \leq \Delta$. Let $D = \sum_{i=1}^{k}(2x_i)^2$. We define a sequence of independent random variables $\{X_i\}_{i=1}^{k}$ as follows:

$$X_i = \begin{cases} -x_i & \text{, with } p = \frac{1}{2} \\ x_i & \text{, with } p = \frac{1}{2} \end{cases}$$

Then $|d_{B_1 \cap N}(v) - d_{B_2 \cap N}(v)| = |\sum_{i=1}^{k} X_i|$, and

$$\Pr(|\sum_{i=1}^{k} X_i| > \frac{1}{10}\sqrt{D}\ln\Delta) < \frac{1}{4n}$$

since $\Delta \geq \frac{n}{6}$ is large enough. Our final observations are that $\sqrt{D} \leq 2\Delta 3/4$ and that $d_{B-N}(v) \leq 10\sqrt{\Delta}$, which together with the above imply our claim. □

5.3 Case 3: $2\Delta <$ def$(B^*) < \Delta^{12/10}$

We present an algorithm to color with Δ colors the edges of a multi-Vizing reduction $G^* = (B^* + s_1, E^*)$ of deficiency $2\Delta < \text{def}(B^*) < \Delta^{12/10}$ where $0 \leq d_{B^*}(s_1) = \text{def}_{s_1}(B^*) \leq \frac{1}{2}\Delta$. As usual, we assume that G^* has large maximum degree Δ, that $|B^* + s_1| < 6\Delta$ and that a modified split partition $(B_1^*, B_2^* + s_1)$ of $B^* + s_1$ is provided. Since there is just one small vertex, we simplify the notation by denoting s_1 by s. We will actually consider only graphs of deficiency greater than $\frac{5}{2}\Delta$ if $|B^*|$ is odd. We will leave the discussion on how to color graphs with odd $|B^*|$ and deficiency at most $\frac{5}{2}\Delta$ to the remaining case 4, which we discuss in chapter 6.

The approach we take in our edge coloring algorithm is similar to the one we took in the medium deficiency case: after defining an initial marking that is proper over initial matchings $M_1', ..., M_{\Delta_1}'$, we modify the marking to prepare it for patching through which we obtain the final matchings $M_1, ..., M_{\Delta_1}$. We must overcome two new obstacles however. First, we must use more careful techniques when modifying the marking to prepare the iitial matchings for patching because little deficiency is available. Second, we must worry about the multiple edges of a multi-Vizing reduction during patching in both coloring passes. In this case, we set Δ_1 to be the smallest even (odd) integer greater than or equal to $\frac{1}{2}\Delta + \Delta^{3/4}\ln\Delta$ if Δ is even (odd) and Δ_2 is the smallest even integer greater than or equal to $\frac{1}{2}\Delta^{19/20}$. Let $\delta = \Delta_1 - \frac{1}{2}\Delta$.

5.3.1 The first coloring pass

In the first coloring pass, we will attempt to construct disjoint matchings $M_1, ..., M_{\Delta_1}$ such that $F = G^* - \cup_{i=1}^{\Delta_1} M_i'$ is a reduction of G^* and $\cup_{i=1}^{\Delta_1} M_i$

contains all edges in $(E(B_1^*) - E(R_1)) \cup (E(B_2^* + s) - E(R_2))$ where R_1 and R_2 are reject subgraphs of B_1^* and B_2^*, respectively, of maximum degree less than $\Delta^{9/10}$ such that $|E(R_1)| = |E(R_2)| < \frac{1}{8}\Delta^{19/10}$. We will insist that

$$d_{B^*}^F(v) \geq \frac{1}{16}\Delta - \delta \text{ for all } v \in B^* \tag{5.1}$$

as it is necessary for patching in either coloring pass. In addition, if $|B^*|$ is odd, enough deficiency must remain in B_1^* for the reject coloring pass

$$\operatorname{def}(B_1^*) - m^{\Delta_1}(B_1^*) \geq \frac{3}{4}\Delta - \frac{1}{8}\delta \tag{5.2}$$

where $m^{\Delta_1}(B_1^*) = \sum_{i=1}^{\Delta_1} \sum_{v \in B_1^*} m(v, i)$. If we don't succeed in constructing $M_1, ..., M_{\Delta_1}$ as desired, we will construct a fail pair (X, Y) in (B_1, B_2). We begin the construction of $M_1, ..., M_{\Delta_1}$ with an initial coloring and an initial marking.

An initial coloring

We observe that $\Delta(B_1^*), \Delta(B_2^*+s_1) < \frac{1}{2}\Delta+\frac{1}{4}\delta < \Delta_1-\sqrt{\Delta}$, implying that we can apply Fournier's multigraph algorithm to Δ_1 edge color B_1^* and $B_2^* + s$. So, we initially construct matchings $M_1', ..., M_{\Delta_1}'$, balanced in B_1^* and in B_2^*, so that $\cup_{i=1}^{\Delta_1} M_i' = E(B_1^*) \cup E(B_2^* + s)$, as follows:

1. We color the edges of $E(B_1^*)$ with Δ_1 colors and, using the procedure from 3.2.2, we obtain the matchings $M_1^1, ..., M_{\Delta_1}^1$, balanced in B_1^*, such that $\cup_{i=1}^{\Delta_1} M_i^1 = E(B_1^*)$. We similarly construct $M_1^2, ..., M_{\Delta_1}^2$, balanced in B_2^* and covering $E(B_2^* + s)$. Then, for every $i = 1, ..., \Delta_1$, we set $M_i' = M_i^1 \cup M_i^2$.

Note that $0 \leq |M_i^1| - |M_i^2 \cap E(B_2^*)| \leq 1$ for every $i = 1, ..., \Delta_1$, since $0 \leq c_B = |E(B_1^*)| - |E(B_2^*)| < \frac{1}{2}\Delta < \Delta_1$. Actually, $|M_i^1| - |M_i^2 \cap E(B_2^*)| = 1$ for exactly c_B indices i.

An initial marking

We define an initial marking of the vertices that is proper over the initial matchings $M_1', ..., M_{\Delta_1}'$. (A marking is proper if $m(v, i) = 1$ implies that M_i' misses v, for all i and v.) Since $\cup_{i=1}^{\Delta_1} M_i$ is supposed to contain c_B more big edges in B_1^* than in B_2^*, we must define a marking that satisfies

$$m^{\Delta_1}(B_2^*) - m^{\Delta_1}(B_1^*) = 2c_B \text{ if } |B^*| \text{ is even} \tag{5.3}$$
$$m^{\Delta_1}(B_2^*) - m^{\Delta_1}(B_1^*) = 2c_B - \Delta_1 \text{ if } |B^*| \text{ is odd} \tag{5.4}$$

We require that all edges in $E(B_2^*, s)$ belong to $\cup_{i=1}^{\Delta_1} M_i'$, so the marking is also required to satisfy

$$m_s(v,i) = 1 \text{ for every } i = 1, ..., \Delta_1 \text{ and } v \in B^* \text{ such that } (s,v) \in M_i' \tag{5.5}$$

Finally, in order for our patching techniques to work, we must mark large deficiency vertices in sufficiently many matchings, and, in order to limit the maximum degree of the reject graphs R_1 and R_2, we must insure that no vertex v is not marked in much more than $\frac{1}{2}\text{def}(v)$ matchings. More precisely, we insist that

$$\max\{0, \frac{9}{16}\Delta - d_{B^*}(v)\} \leq m_r^{\Delta_1}(v) \leq \min\{\frac{1}{2}\text{def}_r(v) + \frac{1}{2}\delta, \text{def}_r(v)\} \tag{5.6}$$

for every $v \in B^*$, and, if $|B^*|$ is even, we additionally require the technical condition

$$m^{\Delta_1}(b_1) \geq c_B \tag{5.7}$$

In order to define a marking that satisfies conditions 5.2 - 5.7, we find it useful to first assign targets $t(v)$ to every $v \in B^*$, where $t(v)$ is the number of matchings $M_1, ..., M_{\Delta_1}$ which are going to miss v:

2.1 If $c_B \geq c_S$, we define $t(v)$ to be $\lfloor\frac{1}{2}\text{def}_r(v)\rfloor$ or $\lceil\frac{1}{2}\text{def}_r(v)\rceil$ for every $v \in B_2^*$, so that $t(B_2^*) = \sum_{v \in B_2^*} t(v) = \lceil\frac{1}{2}\text{def}_r(B_2^*)\rceil$. For every $v \in B_1^*$, we define $t(v)$ so that $\max\{0, \frac{9}{16}\Delta - d_{B^*}(v)\} \leq t(v) \leq \min\{\frac{1}{2}(\text{def}_r(v) + \delta), \text{def}_r(v)\}$ and $t(B_1^*) = \lceil\frac{1}{2}\text{def}_r(B_1^*) - (c_B - c_S)\rceil$ if $|B^*|$ is even, or $t(B_1^*) = \lceil\frac{1}{2}\text{def}_r(B_1^*) + \delta - (c_B - c_S)\rceil$ if $|B^*|$ is odd.

If $c_B < c_S$ (in which case $|B^*|$ must be even), for every $v \in B_2^*$, we define $t(v)$ so that $\max\{0, \frac{9}{16}\Delta - d_{B^*}(v)\} \leq t(v) \leq \lceil\frac{1}{2}\text{def}_r(v)\rceil$ and $t(B_2^*) = \lceil\frac{1}{2}\text{def}_r(B_2^*) - (c_S - c_B)\rceil$; we additionally insist that $t(b_1) \geq c_B$. For every $v \in B_1^*$, we define $t(v)$ to be $\lfloor\frac{1}{2}\text{def}_r(v)\rfloor$ or $\lceil\frac{1}{2}\text{def}(v)\rceil$ and $t(B_1^*) = \lceil\frac{1}{2}\text{def}_r(B_1^*)\rceil$.

Before we show that the described target assignment is feasible, we note that if we defined a marking such that $m_r^{\Delta_1}(v) = t(v)$ and $m_s^{\Delta_1}(v) = \mathrm{def}_s(v)$ for all $v \in B^*$, then conditions 5.2, 5.3, 5.4, 5.6 and 5.7 would follow.

To prove that the target assignments are feasible, we define W_1 and W_2 to be the sets of vertices v in B_1^* and B_2^*, respectively, such that $\mathrm{def}(v) \geq \frac{7}{16}\Delta$, and we show:

Claim 18 *Suppose $|B^*|$ is even and def(B^*) $> 2\Delta$ or $|B^*|$ is odd and def(B^*) $> \frac{5}{2}\Delta$. If $c_B \geq c_S$,*

$$\sum_{v \in W_1} (\frac{9}{16}\Delta - d_{B^*}(v)) \leq \frac{1}{2} \mathit{def}_r(B_1^*) - (c_B - c_S) \tag{5.8}$$

and, if $c_S > c_B$ and $c_B \leq \frac{9}{16}\Delta - d_{B^*}(b_1)$,

$$\sum_{v \in W_2} (\frac{9}{16}\Delta - d_{B^*}(v)) \leq \frac{1}{2} \mathit{def}_r(B_2^*) - (c_S - c_B) \tag{5.9}$$

and, if $c_S > c_B$ and $c_B > \frac{9}{16}\Delta - d_{B^*}(b_1)$,

$$\sum_{v \in W_2 - b_1} (\frac{9}{16}\Delta - d_{B^*}(v)) \leq \frac{1}{2} \mathit{def}_r(B_2^*) - c_S \tag{5.10}$$

Proof: We first prove 5.8. Since $\frac{1}{2}\mathrm{def}_r(B_1^*) - (c_B - c_S) \geq \frac{1}{2}\mathrm{def}(B_1^*) - c_B$, it is enough to show $\sum_{v \in W_1} (\frac{9}{16}\Delta - d_B^*(v)) \leq \frac{1}{2}\mathrm{def}(B_1^*) - c_B$.

If $|B^*|$ is even then $\mathrm{def}(B_1^*) > \frac{3}{4}\Delta$ and $c_B < \frac{1}{4}\Delta$. It follows that $\frac{1}{2}\mathrm{def}(B_1^*) - c_B \geq \frac{3}{8}\Delta - \frac{1}{4}\Delta \geq \frac{1}{8}\Delta \geq \sum_{v \in W_1}(\frac{9}{16}\Delta - d_{B^*}(v))$ if $|W_1| \leq 2$. If $|W_1| \geq 2$, then the following holds: $\frac{1}{2}\mathrm{def}(B_1^*) - c_B \geq \frac{7}{32}\Delta|W_1| - \frac{1}{4}\Delta > \frac{1}{16}\Delta|W_1| > \sum_{v \in W_1}(\frac{9}{16}\Delta - d_{B^*}(v))$.

If $|B^*|$ is odd then $\mathrm{def}(B_1^*) > \frac{5}{4}\Delta$ and $c_B < \frac{1}{2}\Delta$. It follows that $\frac{1}{2}\mathrm{def}(B_1^*) - c_B > \frac{5}{8}\Delta - \frac{1}{2}\Delta \geq \frac{1}{8}\Delta > \sum_{v \in W_1}(\frac{9}{16}\Delta - d_{B^*}(v))$ if $|W_1| \leq 2$. If $|W_1| \geq 4$, then the following holds: $\frac{1}{2}\mathrm{def}(B_1^*) - c_B \geq \frac{7}{32}\Delta|W_1| - \frac{1}{2}\Delta > \frac{1}{16}\Delta|W_1| > \sum_{v \in W_1}(\frac{9}{16}\Delta - d_{B^*}(v))$. Finally, if $W_1 = \{b_1, b_2, b_3\}$, inequality 5.8 is equivalent to $\frac{3}{16}\Delta + c_B \leq \frac{1}{2}(d_{B^*}(b_1) + d_{B^*}(b_2) + d_{B^*}(b_3))$, which in turn is equivalent to $\frac{11}{16}\Delta \leq \frac{12}{16}\Delta$.

If $c_S > c_B$ and $c_B \leq \frac{9}{16}\Delta - d_{B^*}(b_1)$, implying that $|B^*|$ is even, then $\mathrm{def}(B_2^*) > \Delta$ and $\mathrm{def}_r(B_2^*) > \frac{3}{4}\Delta - x$, where $x = \frac{1}{2}(\mathrm{def}_s(B_2^*) - \mathrm{def}_s(B_1^*)) < \frac{1}{8}\delta$. It follows that $\frac{1}{2}\mathrm{def}_r(B_2^*) - (c_S - c_B) \geq \frac{3}{8}\Delta - \frac{1}{2}x - \frac{1}{4}\Delta > \frac{1}{8}\Delta - \frac{1}{8}\delta > \frac{9}{16}\Delta - d_{B^*}(b)$ and 5.9 is true if $W_2 \leq 1$. If $|W_2| \geq 2$ then $\frac{1}{2}\mathrm{def}_r(B_2^*) - (c_S - c_B) \geq \frac{7}{32}\Delta|W_2| - \frac{1}{4}\Delta > \frac{1}{16}\Delta|W_2| \geq \sum_{v\in W_2}(\frac{9}{16}\Delta - d_{B^*}(v))$.

Finally, if $c_S > c_B$ and $c_B > \frac{9}{16}\Delta - d_{B^*}(b_1)$, a similar argument shows 5.10.

□

Once the targets are assigned, we define in the following step the actual marking so that $m_r^{\Delta_1}(v) = t(v)$ for every $v \in B^*$ and 5.5 is satisfied:

2.2 We set $m_s(v, i) = 1$ for every $i = 1, ..., \Delta_1$ and $v \in B^*$ such that $(s, v) \in M_i'$.

2.3 For every $v \in B^*$, we pick a set $\mathcal{M}_v$ of $t(v)$ matchings among $M_1', ..., M_{\Delta_1}'$ missing v and we set $m(v, i) = 1$ for every $M_i' \in \mathcal{M}_v$. We then equalize the marking using the equalizing procedure from 3.2.3.

Note that, while the equalizing procedure may modify the matchings $M_1', ...,$ M_{Δ_1}', they are still balanced in B_1^* and in B_2^* and $\cup_{i=1}^{\Delta_1} M_i' = E(B_1^*) \cup E(B_2^* + s)$. Let $n(B_1^*, i)$ and $n(B_2^*, i)$ be the numbers of vertices in B_1^* and B_2^*, respectively, that are not marked in and are missed by M_i'.

Claim 19 *After step* **2.3**, *the following hold for every* $1 \leq i, j \leq \Delta_1$ *and* $k = 1, 2$:

(a) $|n(B_k^*, i) - n(B_k^*, j)| \leq 2$

(b) $|n(B_1^*, i) - n(B_2^*, j)| \leq 3$

(c) $\frac{3}{2}\delta < n(B_k^*, i) < \frac{27}{4}\delta$

(d) $m(B_k^*, i) \leq 2\Delta^{1/5}$

Proof: **(a)** follows from $n(B_k^*, i) = |B_k^*| - (2|M_i^k| + m(B_k^*, i))$ and because the matchings $M_1^k, ..., M_{\Delta_1}^k$ are balanced in B_k^* and the marking is equalized over the matchings. **(b)** follows from **(a)** and because $\sum_{i=1}^{\Delta_1} n(B_1^*, i) = \sum_{i=1}^{\Delta_1} n(B_2^*, i)$. We now prove that **(c)** holds for $k = 1$; a symmetric argument does the job for $k = 2$. We observe that $\sum_{i=1}^{\Delta_1} n(B_1^*, i) = \sum_{v\in B_1^*}(\Delta_1 -$

$d_{B_1^*}(v) - m^{\Delta_1}(v))$ and also $|d_{B_1^*}(v) - \frac{1}{2}d_{B^*}(v)| < \frac{1}{8}\delta$ and $|m^{\Delta_1}(B_1^*) - \frac{1}{2}\mathrm{def}(B_1^*)| < \frac{1}{4}\Delta$. It follows that $\frac{7}{8}\delta|B_1^*| - \frac{1}{4}\Delta < \sum_{i=1}^{\Delta_1} n(B_1^*, i) < \frac{9}{8}\delta|B_1^*| + \frac{1}{4}\Delta$. **(c)** then follows from **(a)** and $\frac{1}{2}\Delta \leq |B_1^*| \leq 3\Delta$. □

Preparing the marking and the matchings for patching

We now move some marks between matchings $M_1', ..., M_{\Delta_1}'$, and, in the process, we modify the initial matchings until every matching M_i' misses the same number of unmarked big vertices in B_1^* and B_2^*. More precisely, we will obtain a marking that is proper over the modified matchings $M_1', ..., M_{\Delta_1}'$ so that $\cup_{i=1}^{\Delta_1} M_i' = (E(B_1^*) - E(R_1)) \cup (E(B_2^* + s) - E(R_2))$ where:

(i) $m_r^{\Delta_1}(v) = t(v)$ and $m_s^{\Delta_1}(v) = \mathrm{def}_s(v)$ for all $v \in B^*$.

(ii) $m(B_1^*, i), m(B_2^*, i) < 2\Delta^{1/5}$ and $\delta < n(B_1^*, i) = n(B_2^*, i) < 8\delta$ for every $i = 1, ..., \Delta_1$.

(iii) R_1 and R_2 are reject subgraphs of B_1^* and B_2^*, respectively, of maximum degree less than $\Delta^{9/10}$ and $|E(R_1)| = |E(R_2)| < 2\Delta$.

We define two partitions, (I_e^1, I_o^1) and (I_e^2, I_o^2), of $I = \{1, ..., \Delta_1\}$ as follows: $i \in I_e^k$ if and only if $m(B_k^*, i)$ is even. For example, $i \in I_o^2$ if and only if M_i^2 contains an odd number of marks on vertices in B_2^*. In order to insure $n(B_1^*, i) = n(B_2^*, i)$ for all $i = 1, ..., \Delta_1$, we must reorder the matchings $M_1^1, ..., M_{\Delta_1}^1$ and $M_1^2, ..., M_{\Delta_1}^2$ so that $m(B_1^*, i)$ and $m(B_2^*, i)$ have the same parity if $|B^*|$ is even, and opposite parities if $|B^*|$ is odd. We can do the reordering only if $|I_e^1| = |I_e^2|$ when $|B^*|$ is even or $|I_e^1| = |I_o^2|$ when $|B^*|$ is odd: we call this the **parity condition**. If the parity condition holds, we move directly to reordering the matchings in step **3.4**.

If, however, the parity condition does not hold, and either $|B^*|$ is even and $||I_e^1| - |I_e^2|| = 2d > 0$ or $|B^*|$ is odd and $||I_e^1| - |I_o^2|| = 2d > 0$, we must move a few marks between some matchings: by switching, in $2d$ appropriately chosen matchings, the parity of the number of marks, we will insure the parity condition. We use one of the following procedures to move a mark on some vertex v from matching M' (that misses v) to another matching M:

Simple mark move If M misses v, we just move the mark on v from M' to M.

First edge rejecting mark move If there is an edge $(v, u) \in M$, we reject (v, u), we move the mark on v from M' to M (figure 5.1).

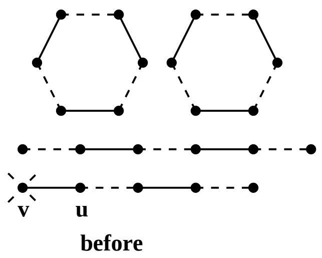

Figure 5.1: A simple mark move on v from M' (dashed) to M (full)

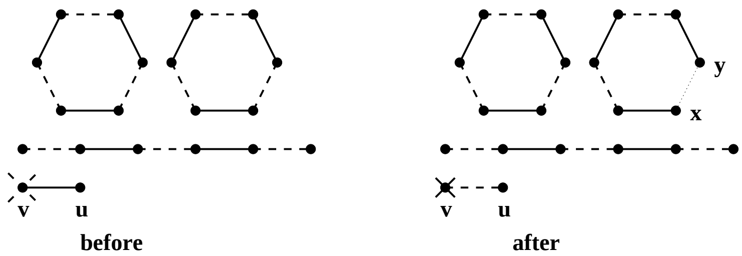

Figure 5.2: A first edge recoloring mark move on v from M' (dashed) to M (full)

First edge recoloring mark move If there is an edge $(v, u) \in M$, u is missed by and not marked in M', we move the mark on v from M' to M, and we pick some edge $(x, y) \in M'$ and we reject it (figure 5.2).

Second edge rejecting mark move If there is an edge $(v, u) \in M$ and there is an edge $(u, w) \in M'$, we reject (u, w), we switch (v, u) from M to M', and we move the mark on v from M to M'.

We observe that in any of the mark moves, the matching M' (the mark "giver") remains of the same size, while M (the mark receiver) may lose an edge to the reject graphs. In order to limit the maximum degree of the reject graph, we must carefully choose our sequence of marks moves, which we do using:

The careful procedure

Let I_+ and I_- be disjoint sets of indices of matchings $\{M_1^k, ..., M_{\Delta_1}^k\}$ such that either

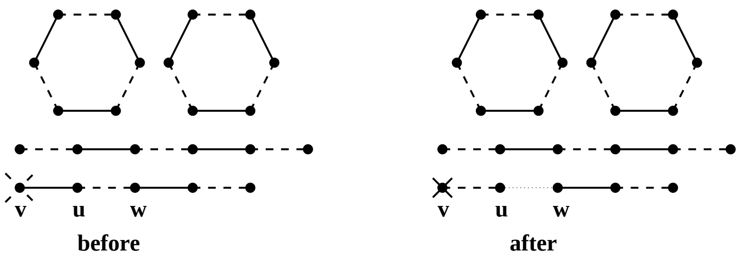

Figure 5.3: A second edge rejecting mark move on v from M' (dashed) to M (full)

(i) $|I_-| \geq \frac{1}{2}\Delta_1 + |I_+|$, or

(ii) $|I_-| \geq |I_+|$ and $m(v,j) = 0$ for every $j \in I_-$ if $m(v,i) = 1$ for some $i \in I_+$ and $v \in B_k^*$

Recursively, for every $i \in I_+$, we pick some $v \in B_k^*$ such that $m(v,i) = 1$, and we move the mark on v to some matching indexed by I_- chosen in one of the following four ways:

1. If there exist $j \in I_-$ such that $m(v,j) = 0$ and M_j misses v, we use a simple mark move on v from M_i to M_j.

If we are not successful in **1**, we set I_* to be the subset of indices in I_- such that $m(v,i) = 0$. We observe that M_i hits v for every $i \in I_*$ and $|I_*| > |I_+|$, which follows from our definitions of I_+ and I_- and because $m^{\Delta_1}(v) < \frac{1}{2}\Delta_1$ for every $v \in B^*$ (see 5.6). Then, we do one of the following:

2. If $|I_*| \leq 2\Delta^{3/4}$, we pick any $j \in I_*$ and we use the first edge rejecting mark move on v from M_i to M_j.

3. If $|I_*| > 2\Delta^{3/4}$ and there exists $j \in I_*$ such that $(v,u) \in M_j$, u is missed by M_i and $m(v,i) = 0$, we use the first edge recoloring mark move on v from M_i to M_j, picking the edge (x,y) so no edge incident to either x or y has been rejected in the previous $\sqrt{\Delta}$ mark moves.

4. If $|I_*| > 2\Delta^{3/4}$ and we fail to find j as desired in step **3**, we pick $j \in I_*$ such that there is $(v,u) \in M_j$ and $(u,w) \in M_i$ and no edges incident to u or w have been rejected in the last $\frac{1}{2}\Delta^{1/4}$ edges mark moves, and we apply the second edge rejecting mark move.

We show that we can choose j in step **4** as desired. At least $2\Delta^{3/4}$ matchings indexed by I_* hit v and because the multiplicity of any edge is at most $\sqrt{\Delta}$, there are at least $2\Delta^{\frac{1}{4}}$ different vertices u such that $(v,u) \in M_j$ for some $j \in I_*$. At most $2\Delta^{1/5}$ such vertices are marked in M_i so there are more than $2\Delta^{1/4} - 2\Delta^{1/5} > \Delta^{1/4}$ vertices u such that $(v,u) \in M_j$ for some $j \in I_*$ and $(u,w) \in M_i$ for some vertex w. Finally, since no more than $\Delta^{1/4}$ edges in M_i have an endpoint incident to an edge rejected in the previous $\frac{1}{2}\Delta^{1/4}$ iterations, there exists $j \in I_*$ as desired in step **4**. We observe that the procedure insures that no vertex is incident to more than $2\Delta^{3/4} + \frac{\Delta_1}{\frac{1}{2}\Delta^{1/4}} < 4\Delta^{3/4}$.

We now describe how we use the careful procedure to insure the parity conditions. If $|B^*|$ is odd, note that either $|I_o^1|+|I_o^2| = \Delta_1+2d$ or $|I_e^1|+|I_e^2| = \Delta_1 + 2d$. In the first case we set $I_+^1 = I_o^1$ and $I_+^2 = I_o^2$, while in the second case we set $I_+^1 = I_e^1$ and $I_+^2 = I_e^2$. If $|B^*|$ is even, either $|I_o^1| + |I_e^2| = \Delta_1 + 2d$ or $|I_e^1| + |I_o^2| = \Delta_1 + 2d$. In the first case we set $I_+^1 = I_o^1$ and $I_+^2 = I_e^2$, while in the second case we set $I_+^1 = I_e^1$ and $I_+^2 = I_o^2$. Whether $|B^*|$ is odd or even, we define $D_1 = |I_+^1| - \frac{1}{2}\Delta_1$ and $D_2 = |I_+^2| - \frac{1}{2}\Delta_1$. Note that while one of D_1 or D_2 may be negative, their sum is always equal to $2d$. We first attempt to move the marks as follows:

3.1 Let I_+^k to be the larger one of I_+^1 and I_+^2, and I_+^l to be the smaller one of the two. We set I_*^k to be a subset of I_+^k of size $d' = \min\{d, \lceil \frac{1}{2}D_1 \rceil\}$ such that $m_r(B_k^*, i) > 0$ (*note subscript*) for every $i \in I_*^k$ and we move d' *real* marks from I_*^k to $I_+^k - I_*^k$ using the careful procedure. (Note that $|I_+^k - I_*^k| > \frac{1}{2}\Delta_1 + |I_*^k|$.) We then set I_*^l to be a subset of I_+^l of size $d - d'$ (either 0 or equal to $\lfloor \frac{1}{2}D_2 \rfloor \geq 1$) such that $m_r(B_l^*, i) > 0$ for every $i \in I_*^l$, and we move $d - d'$ *real* marks from I_*^l to $I_+^l - I_*^l$ using the careful procedure. (Note that $|I_+^l - I_*^l| > \frac{1}{2}\Delta_1 + |I_*^l|$).

This sequence of mark moves fails if there are less than d' indices i in I_+^k such that $m(B_k^*, i) > 0$, or if there are less than $d - d'$ indices i in I_*^l such that $m(B_l^*, i) > 0$. In the first case, $I_+^k = I_e^k$, $m(B_k^*, i) = 1$ for every $i \in I_o^k$ so that $m^{\Delta_1}(B_k^*) < \frac{1}{2}\Delta_1$. Since $m^{\Delta_1}(B^*) > \frac{3}{4}\Delta - \frac{1}{8}\delta$ and $m^{\Delta_1}(B_1^*) - \delta < m^{\Delta_1}(B_2^*)$ (follow from our terget assignments), it follows that $k = 1$. In the second case, a symmetric argument also gives $l = 1$. So, if we fail in step **3.1**, we do one of **3.2** or **3.3** instead, as described below.

3.2 If $|B^*|$ is odd, we first set I_*^1 to be the subset of I_e^1 of all indices such that $m(B_1^*, i) = 2$, and we move $|I_*^1|$ (real) marks from I_*^1 to $I_e^1 - I_*^1$ (note that $|I_e^1 - I_*^1| > \frac{1}{2}\Delta_1 + |I_*^1|$). After this sequence of mark moves, we obtain the modified partition (I_e^1, I_o^1) that satisfies $|I_o^1| = m^{\Delta_1}(B_1^*) < \frac{1}{2}\Delta_1$ so that $m(B_1^*, i) = 1$ for every $i \in I_o^1$ and $m(B_1^*, i) = 0$ for every $i \in I_e^1$. We remark also that $m(B_2^*, i) = 1$ for every $i \in I_o^2$, because the marking is equalized over $M_1^2, ..., M_{\Delta_1}^2$ and $m^{\Delta_1}(B_2^*) \le m^{\Delta_1}(B_1^*) + \frac{1}{2}\Delta$. It also follows that there are at least $d - |I_*^1|$ indices i in I_e^2 such that $m(B_2^*, i) = 0$. Since we defined a marking that satisfies $m^{\Delta_1}(B^*) > \Delta_1$ (easy exercise), there must be at least $\frac{1}{2}(\Delta_1 - |I_o^1| - |I_o^2|) = d - |I_*^1|$ indices i in I_e^2 such that $m(B_2^*, i) = 2$. So, we set I_*^2 to be a subset of I_e^2 of size $d - |I_*^1|$ such that $m(B_2^*, i) = 2$ for every $i \in I_*^2$, we set I_-^2 to be the subset of I_e^2 of $d - |I_*^1|$ indices i such that $m(B_2^*, i) = 0$ and we move $d - |I_*^1|$ *real* marks from I_*^2 to I_-^2.

3.3 If $|B^*|$ is even, we first set I_*^1 to be the subset of I_e^1 of indices i such that $m(B_1^*, i) = 2$ and we move $d_1 = |I_*^1|$ (real) marks from I_*^1 to $I_e^1 - I_*^1$. After this sequence of mark moves, we obtain a modified partition (I_e^1, I_o^1) that satisfies $|I_o^1| = m^{\Delta_1}(B_1^*) < \frac{1}{2}\Delta_1$, so that $m(B_1^*, i) = 1$ for every $i \in I_o^1$ and $m(B_1^*, i) = 0$ for every $i \in I_e^1$. Note that $m(B_2^*, i) = 1$ for every $i \in I_o^2$, because the marking is equalized over B_2^* and $m^{\Delta_1}(B_2^*) = m^{\Delta_1}(B_1^*) + 2c_B < m^{\Delta_1}(B_1^*) + \frac{1}{2}\Delta$. It also follows that $m^{\Delta_1}(B_2^*) = |I_o^1| + 2c_B = |I_o^2| + 2(c_B - (d - d_1))$, so that at most $c_B - (d - d_1)$ indices i in I_e^2 satisfy $m(B_2^*, i) > 0$ ($= 2$, actually). Since $m^{\Delta_1}(b) \ge c_B$ (from 5.7), it follows that there are at least $d - d_1$ indices i in I_o^2 such that $m(b, i) = 1$. (Recall that b is the largest deficiency vertex in B_2^*.) Let d_2 be the number of indices in I_o^2 such that $m_r(B_2^* - b, i) = 1$. Depending on the value of d_2, we do one of the following:

3.3.1 If $d_2 \ge d - d_1$, we set I_*^2 to be the subset of I_o^2 of size $d - d_1$ such that $m_r(B_2^* - b, i) = 1$ for every $i \in I_*^2$, we set I_-^2 to be a subset of I_o^2 of size $d - d_1$ such that $m(b, j) = 1$ for every $j \in I_-^2$ and we move $d - d_1$ *real* marks from I_*^2 to I_-^2 (note that $|I_*^1| = |I_-^1|$ and that $m(B_2^* - b, j) = 0$ for every $j \in I_o^2$).

3.3.2 If $d_2 < d - d_1$, we set I_*^2 to be the subset of I_o^2 of size d_2 such that $m_r(B_2^* - b, i) = 1$ for every $i \in I_*^2$, and we move d_2 *real* marks from I_*^2 to $I_2^o - I_*^2$ (note that $|I_o^2 - I_*^2| > |I_*^2|$ and $m(B_2^* - b, j) = 0$ for every $j \in I_o^2 - I_*^2$). When done, we reset I_*^2 to be the subset of I_e^2 of size $d - d_1 - d_2$ such that $m(B_2^* - b, i) = 2$ for every $i \in I_*^2$. Note that there must be that many such indices because otherwise $\lceil \frac{1}{4}\Delta \rceil + \sqrt{\Delta} \ge m^{\Delta_1}(b) > m^{\Delta_1}(B^* - b) > \lfloor \frac{1}{2}\Delta \rfloor - \sqrt{\Delta}$, a contradiction. We move $d - d_1 - d_2$ marks from I_*^2 to I_o^2 (note that $|I_o^2| > |I_*^2|$ and that $m(B_2^* - b, j) = 0$ for every $j \in I_o^2$). When done, there will be exactly $d - d_1 - d_2$ indices i in I_o^2 with $m_r(B_2^* - b, i) = 1$. Finally, we set I_*^2 to be the subset of these indices and we move $d - d_1 - d_2$ *real* marks from I_*^2 to $I_o^2 - I_*^2$ (note that $|I_o^2 - I_*^2| > |I_*^2|$ and that $m(B_2^* - b, j) = 0$ for every $j \in I_o^2 - I_*^2$.)

When done, the parity conditions are satisfied. In steps **3.1**, **3.2** or **3.3**, we move at most $2d < \Delta_1$ marks and we use the careful procedure at most three times. So the total number of rejected edges on each side of the bipartition is at most Δ_1 but no more than $12\Delta^{3/4} < \frac{1}{2}\Delta^{4/5}$ (for large Δ) is incident to any particular vertex. A closer analysis reveals that no matching has increased in size, no matching has decreased by more than 1 (due to edge rejection), and no matching has lost or gained more than one mark. So, using claim **(b)** of 19, we have that $|n(B_k^*, i) - n(B_k^*, j)| \leq 6$. It now remains to satisfy $n(B_1^*, i) = n(B_2^*, i)$ for all $i = 1, ..., \Delta_1$:

3.4 We reorder the indices of matchings $M_1^1, ..., M_{\Delta_1}^1$ so that $m(B_1^*, i)$ and $m(B_2^*, i)$ have the same parity if $|B^*|$ is even or different parities if $|B^*|$ is odd. We then set $M_i' = M_i^1 \cup M_i^2$ for every $i = 1, ..., \Delta_1$. Note that after reordering the matchings, $|n(B_1^*, i) - n(B_2^*, i)|$ is even and at most 6 for every $i = 1, ..., \Delta_1$.

For every $i = 1, ..., \Delta_1$, we delete $d = \frac{1}{2}|n(B_1^*, i) - n(B_2^*, i)| \leq 3$ edges in $E(B_1^*)$ if $n(B_1^*, i) < n(B_2^*, i)$ or in $E(B_2^*)$ if $n(B_1^*, i) > n(B_2^*, i)$. In each iteration, we pick d edges whose endpoints have not had an incident edge rejected in the previous $\sqrt{\Delta}$ iterations.

Since there can be at most Δ_1 iterations in step **3.4**, no vertex is incident to more than $\sqrt{\Delta}$ rejected edges. Since $m^{\Delta_1}(B_1^*) - m^{\Delta_1}(B_2^*) = 2c_B$ if $|B^*|$ is even or $2c_B - \Delta_1$ if $|B^*|$ is odd, it follows that $|E(R_1)| = |E(R_2)| < 2\Delta$ and that the maximum degree of R_1 and R_2 is at most $\Delta^{4/5}$. Finally, $\delta < n(B_1^*, i) = n(B_2^*, i) < 8\delta$ follows from claim 19.

The patching

For $i = 1, ..., \Delta_1$, we recursively obtain M_i by augmenting the vertex disjoint patches we construct between pairs of unmarked big vertices missed by M_i' in $F = G^* - R_1 - R_2 - M_1 - ... - M_{i-1}$. After each augmentation, we add the edges of M_i' left uncolored by this augmentation to the reject graphs R_1

or R_2. If we fail to construct a patch between two unmarked vertices missed by some M_i', we will show the existence of and construct a fail pair (X, Y) in (B_1, B_2). On the other hand, if we are successfull, the big edges of every matching M_i will miss only the vertices v marked in it (i.e. all v such that $m(v, i) = 1$), and $F = G^* - \cup_{i=1}^{\Delta_1} M_i$ is a reduction as desired.

We now describe the construction of the vertex disjoint patches of M_i' in $F = G^* - R_1 - R_2 - M_1 - ... - M_{i-1}$. We recall that a patch P between unmarked vertices $x \in B_1^*$ and $y \in B_2^*$ missed by M_i' is a path from x to y with edges alternating between $E(F) \cap E(B_1^*, B_2^*)$ and $M_i' \cap E(B)$. For $r = 1, ..., n(B_1^*, i)$, we recursively construct the patch P_r as follows:

4.1 We pick a pair of unmarked vertices x_r and y_r on opposite sides of the bipartition so that x_r and y_r are missed by M_i' and have not yet been patched. We pick y_r so it belongs to B. In choosing x_r we always give priority to unmarked, missed vertices in $B^* - B$.

Since $|B^* - B| < 2\Delta^{1/5} < n(B_1^*, i)$, there are more than $|B^* - B|$ vertices in B that are not marked and are missed by M_i' - our choice of the patch endpoints x_r and y_r is thus feasible.

4.2 We define **unavailable** and **usable** vertices. We call $v \in B$ **unavailable** if it is an internal vertex of any patch $P_1, ..., P_{r-1}$ or of any patch constructed for one of the previous $8\lceil\Delta^{1/10}\rceil$ matchings $(M_{i-1}, ..., M_{i-8\lceil\Delta^{1/10}\rceil})$. We note that if a vertex is *not* unavailable, then no edge incident to it has been rejected while obtaining one of the previous $8\lceil\Delta^{1/10}\rceil$ matchings. We call $v \in B$ **usable** if $v = y_r$ or $(v, u) \in M_i'$ and neither v nor u is unavailable. If $x_r \in B^* - B$, we also define Y^0**-unavailable** and Y^0**-usable** vertices. We call $v \in B$ Y^0**-unavailable** if it is an internal vertex of any patch $P_1, ...P_{r-1}$, or if it is an endpoint of the second edge of a patch out of a vertex in $B^* - B$ constructed for one of the previous $8\lceil\Delta^{1/10}\rceil$ matchings. We call $v \in B$ is Y^0-usable if $v = y_r$ or $(v, u) \in M_i'$ and neither u nor v are Y^0-unavailable.

4.3 We recursively build the sets X^l and Y^l for $0 \le l \le 6\lceil\Delta^{1/20}\rceil$ as follows:

$X^0 = \{x_r\}$,

if $x^r \in B^* - B$, $Y^0 = \{v \in B : v$ is Y^0-usable and $(x_r, v) \in E(F) \cap E(B_1^*, B_2^*)\}$,

if $x^r \in B$, $Y^0 = \{v \in B : v$ is usable and $(x_r, v) \in E(F) \cap E(B_1^*, B_2^*)\}$,
$X^l = \{v \in B : \exists u \in Y^{l-1}$ such that $(u,v) \in M_i'\}$
$Y^l = \{v \in B : v$ is usable and $\exists u \in X^l$ such that $(u,v) \in E(F) \cap E(B_1, B_2)\}$.

If $x_r \in B^* - B$, then at most $6\lceil\Delta^{1/20}\rceil 8\Delta^{1/10}$ vertices in B belong to $P_1, ..., P_{r-1}$ and, as such, are Y^0-unavailable. Furthermore, at most $8\lceil\Delta^{1/10}\rceil$ $|B^* - B| < 8\lceil\Delta^{1/10}\rceil 2\Delta^{1/5}$ vertices are endpoints of the second edge of a patch out of a vertex in $B^* - B$ constructed for one of the previous $8\lceil\Delta^{1/10}\rceil$ matchings, and as such, are Y^0-unavailable. So, at most $\Delta^{2/5}$ vertices are not Y^0-usable.

4.4 If $y_r \in Y^j$ for some $0 \leq j \leq 6\lceil\Delta^{1/20}\rceil$, we construct the patch defined by the sequence of vertices $x_r, y^0, x^1, y^1, ..., y^{j-1}, x^j, y_r$ where $x^l \in X^l$, $y^l \in Y^l$, (x_r, y^0), (x^j, y_r) and (x^l, y^l) belong to $E(F) \cap E(B_1^*, B_2^*)$ and $(y^l, x^{l+1}) \in M_i'$.

Note that each patch contains the same number of edges from $E(B_1)$ and from $E(B_2)$; so when done, $|E(R_1)| = |E(R_2)| < 24\Delta_1\delta\Delta^{1/20} < \frac{1}{10}\Delta^{19/10}$. Furthermore, no vertex is incident to more than $\frac{\Delta_1}{8\Delta^{1/10}} < \frac{1}{8}\Delta^{9/10}$ edges rejected in this step.

4.5 If there is no Y^j containing y_r, then we pick the smallest $j \geq 1$ such that $|Y^j| \leq |X^j| + \frac{1}{2}\Delta^{19/20}$. If $X^j \subset B_1$ then the pair (X, Y), defined by $X = \{v \in X^j \cap B : d^F_{B_2}(v) > \frac{1}{2}\Delta - \Delta^{9/10}\}$ and $Y = N_F(X) \cap B$, form a fail pair.

If $F = G^* - M_1 - ... - M_{i-1}$, then $d^F_{B^*}(v) = \Delta - \Delta_1 - (\mathrm{def}(v) - m^{\Delta_1}(v)) > \frac{1}{16}\Delta - \Delta^{9/10}$. Since $\Delta(R_1)$ and $\Delta(R_2)$ are less than $\Delta^{9/10}$, it follows that $d^F_{B_2}(v) > \frac{1}{16}\Delta - \frac{1}{8}\Delta^{19/20}$ for every $v \in B_1^*$, and similarly, $d^F_{B_1}(v) > \frac{1}{16}\Delta - \frac{1}{8}\Delta^{19/20}$ for every $v \in B_2^*$.

Claim 20 *(X, Y) forms a fail pair in (B_1, B_2).*

Proof: To simplify notation, let $x = x_r \in B_1^*$, so that $y = y_r \in B_2$. Let us first set an upper bound on the sizes of F_1 and F_2, the unavailable vertices in B_1 and B_2, respectively. Since a patch contains at most $6\lceil\Delta^{1/20}\rceil$ vertices in B_1, and since there are at most 8δ patches per matching, it follows that

$$|F_1| < 8\lfloor\Delta^{1/10}\rfloor 8\delta 6\lceil\Delta^{1/20}\rceil < \frac{1}{8}\Delta^{19/20}$$

for large enough Δ. A symmetrical argument gives $|F_2| < \frac{1}{8}\Delta^{19/20}$.

If $x \in B^* - B$, there are at most $\Delta^{2/5}$ Y^0-unavailable vertices. Since $d^F_{B_2}(x) \geq \frac{1}{16}\Delta - \frac{1}{8}\Delta^{19/20}$ and because the edge multiplicity is at most $\sqrt{\Delta}$, $N^F_{B_2}(x) \geq \frac{1}{20}\sqrt{\Delta}$ and $|X^1| = |Y^0| \geq N^F_{B_2}(x) - \Delta^{2/5} > 0$. So, there must exist a vertex $v \in X^1$, which implies that $|X^2| = |Y^1| \geq d^F_{B_1}(v) - |F_1| > \frac{1}{16}\Delta - \frac{1}{8}\Delta^{19/20}$. If $x \in B$ then $|X^2| = |Y^1| \geq \frac{1}{16}\Delta - \frac{1}{8}\Delta^{19/20}$ easily holds.

Suppose that $|Y^l| > |X^l| + \frac{1}{2}\Delta^{19/20}$ for all $2 \leq l \leq \lceil 6\Delta^{1/20}\rceil - 2$. Then, $|X^{6\lceil\Delta^{1/20}\rceil-2}| > 3\Delta \geq |B_1|$, a contradiction.

So we must have $|Y^j| \leq |X^j| + \frac{1}{2}\Delta^{19/20}$ for some j between 1 and $\lceil 6\Delta^{1/20}\rceil - 2$, and we pick the minimum j satisfying this property. Let E_1 and E_2 be subsets of B_1 and B_2, respectively, missed by M_i'. Clearly, $|E_k| \leq n(B^*_k, i) + m(B^*_k, i) < 8\delta + 2 < 10\delta$. If j is even (and $X^j \subset B_1$ and $Y^j \subset B_2$), we define $Y = Y^j \cup E_2 \cup F_2$; if j is odd (and $X^j \subset B_2$ and $Y^j \subset B_1$), we define $Y = Y^j \cup E_1 \cup F_1$. Let $X = \{v \in X^j : d^F_Y(v) > \frac{1}{2}\Delta - \Delta^{19/20}\}$. (X, Y) forms a fail pair by the following three properties:

i. $|Y| \leq |X| + \Delta^{19/20}$.
Proof: We assume $X \subset B_1$ and $Y \subset B_2$; a symmetric argument proves the statement when $X \subset B_2$ and $Y \subset B_1$. We note that $|Y| \leq |Y^j| + |E_2| + |F_2| \leq |X^j| + \frac{1}{2}\Delta^{19/20} + 10\delta + \frac{1}{8}\Delta^{19/20} < |X^j| + \frac{3}{4}\Delta^{19/20}$. Finally, we argue that $|X^j| < |X| + \frac{1}{4}\Delta^{19/12}$ by showing that $X^j - X \subset L$ where $L = v_1 + \{v \in B : \mathrm{def}(v) > 2\sqrt{\Delta}\}$ and $|L| < \sqrt{\Delta}$): if $v \in B_1 - L$ then $d^F_{B_2}(v) \geq d^F_{B^*}(v) - \Delta^{3/5} \geq \Delta(F) - 2\sqrt{\Delta} - \Delta^{3/5} > \frac{1}{2}\Delta - \frac{1}{8}\Delta^{19/20}$, implying $v \notin X^j - X$.

ii. For all $v \in X$: $d_Y(v) > \frac{1}{2}\Delta - \Delta^{19/20}$.
Proof: Assuming $X \subset B_1$, $d_Y(v) > d^F_Y(v) = d^F_{B_2}(v) > \frac{1}{2}\Delta - \frac{1}{8}\Delta^{19/20}$ by the definition of X. A symmetric argument applies if $X \subset B_2$.

iii. If $X \subset B_k$ then $|B_k| - |X| > \frac{1}{4}\Delta - \Delta^{19/20}$.
Proof: If $X \subset B_1$ then y must belong to $B_2 - N^F(X^j)$ which implies that $d^F_X(y) = 0$. Since $y \in B_2$ then $d^F_{B_1}(y) > \frac{1}{4}\Delta - \frac{1}{8}\Delta^{19/20}$. It follows that $|B_1 - X| > \frac{1}{4}\Delta - \frac{1}{8}\Delta^{19/20}$. If $X \subset B_2$, we first note that

$|N^F_{B_1}(y)| > \frac{1}{4}\Delta - \frac{1}{8}\Delta^{19/20}$ (since $y \in B_2$). Let Z be the subset of $N^F_{B_2}(y)$ of vertices incident to some edge of $M'_i \cap E(B^1)$. It is easy to check that $N^F_{B_2}(y) - Z$ contains only vertices that are missed by M'_i, that belong to $B^* - B$ or that are forbidden. Since there is fewer than $m(B^*_1, i) + n(B^*_1, i) + |F_1| + |B^* - B| < \Delta^{19/20}$ such vertices, Z must be non-empty. Let (u, v) be an edge of M'_i in $E(B^*_1)$ with $v \in Z$. Since $d^F_{B_2}(u) > \frac{1}{4}\Delta - \frac{1}{8}\Delta^{19/20}$ and $d^F_X(u) = 0$ (because otherwise a patch t y through a vertex in X and vertices u and v would have been possible), $|B_2 - X| > \frac{1}{4}\Delta - \frac{1}{8}\Delta^{19/20}$.

□

5.3.2 The reject coloring pass

Once we complete the first pass and delete the matchings $M_1, ..., M_{\Delta_1}$ from G^*, we obtain the reduction $F = G^* - \cup_{i=1}^{\Delta_1} M_i = R_1 + H_1 + R_2$ where $H_1 \subset E(B^*_1 \cup S^*_1, B^*_2 \cup S^*_2)$ and R_1 and R_2 are reject graphs in B^*_1 and B^*_2, respectively, of maximum deggree less than $\Delta^{9/10}$ such that $E(R^*_1)| = |E(R^*_2)| < \frac{1}{8}\Delta^{19/10}$. In addition, we know that

$$d^F_{B^*}(v) \geq \frac{1}{16}\Delta - \delta \text{ for all } v \in B^* \tag{5.11}$$

from 5.1 and there is "plenty" of deficiency remaining in B^*_1 (5.2)

$$\text{def}(B^*_1) - m^{\Delta_1}(B^*_1) \geq \frac{3}{4}\Delta - \frac{1}{8}\delta \tag{5.12}$$

In the reject coloring pass, we will attempt to construct the disjoint matchings $M_{\Delta_1+1}, ..., M_{\Delta_1+\Delta_2}$ in F such that $K = F - \cup_{i=1}^{\Delta_2} M_{\Delta_1+i}$ is a reduction of F, and thus of G^*.

The initial coloring

We first construct an initial coloring:

1. We construct initial matchings $M'_1, ..., M'_{\frac{1}{2}\Delta_2}$ balanced in B^*_1 and in B^*_2 and covering $R_1 \cup R_2$ using Fournier's edge coloring algorithm and our balancing procedure (see 3.2.2).

Since $\Delta_2 \geq \frac{1}{2}\Delta^{19/20}$ and $|E(R_1)| = |E(R_2)| < \frac{1}{8}\Delta^{19/20}$, it follows that $|M'_i \cap E(B^*_1)| = |M'_i \cap E(B^*_2)| < \lceil\frac{1}{4}\Delta^{19/20}\rceil$.

The marking

We split each M'_i into M''_{Δ_1+2i-1} and M''_{Δ_1+2i} and obtain disjoint matchings $M''_{\Delta_1+1}, ..., M''_{\Delta_1+\Delta_2}$ covering $E(R_1) \cup E(R_2)$ and to define a proper marking over those matchings such that, for every $i = 1, ..., \Delta_2$:

(i) $|M''_{\Delta_1+i} \cap E(B^*_1)| = |M''_{\Delta_1+i} \cap E(B^*_1)| < \lceil\frac{1}{4}\Delta^{19/20}\rceil$,

(ii) $m(B^*_1, \Delta_1 + i) = 1$ and $m(B^*, \Delta_1 + i = 0$ for all $i = 1, ..., \Delta_2$,

(iii) $n(B_1, \Delta_1 + i) = n(B^*_1, \Delta_1 + i) = n(B^*_2, \Delta_1 + i) = n(B_2, \Delta_1 + i)$.

The last condition implies that in no matching M''_{Δ_1+i} a vertex v in $B^* - B$ is missed and not marked. So, we will only need to patch vertices in B in the patching step.

Let us denote $\sum_{j=1}^{i} m_u(v, \Delta_1 + j)$ by $m_u^{\Delta_1+i}(v)$ for $i = 1, ..., \Delta_2$, $v \in B^*$ and $u = r$ or $u = s$. We define the marking and we construct the matchings by repeating the following for $i = 1, 2, ..., \frac{1}{2}\Delta_2$:

2.1 We set $m_u(v, \Delta_1+2i-1) = 1$ for some $v \in B^*_1$ such that $\mathrm{def}_u(v) > m_u(v)$ where $u = r$ or $u = s$. Then, we set $m_{u'}(v', \Delta_1 + 2i) = 1$ for some $v' \in B^*_1$ such that $\mathrm{def}_{u'}(v') > m_{u'}^{\Delta_1+2i-1}(v')$ where $u' = r$ or $u = s$ and $(v, v') \notin M'_i$.

We can choose v and v' as desired because 5.2 implies that at least 3 vertices have some remaining deficiency.

2.2.1 We remove from M'_i the edge incident to v, if any, and we add it to M''_{Δ_1+2i}. Note that this edge is not incident to v'. If v' is hit by M'_i, let v'' be the vertex v' is matched with.

2.2.2 We now construct M''_{2i-1} from M'_i so that no vertex in B^*-B is missed and unmarked. Let U be the set of unmarked vertices in B^*-B missed by M'_i. We construct a matching M in $(F - v - v' - v'') \cap (B^*_1, B^*_2) - M''_{\Delta_1+1} - ... - M''_{\Delta_1+2i-2}$ such that every $v \in U$ is an endpoint of some edge in M. (We esily obtain this matching because the neighborhood of every $v \in B^* - B$ is at least $\frac{1}{10}\sqrt{\Delta} > |B^* - B|$.) We remove from

M_i' edges incident to M and we add them to M''_{Δ_1+2i}. We remove additional edges not incident to vertices in W_2 from $M_i' \cap B_1^*$ or $M_i' \cap B_2^*$ and we add them to M''_{Δ_1+2i} so $|M''_{\Delta_1+2i} \cap B_1^*| = |M''_{\Delta_1+2i} \cap B_2^*| < \Delta^{1/5}$. We then set $M''_{\Delta_1+2i-1} = M_i' \cup M$.

2.2.3 We now finish the construction of M''_{Δ_1+2i}, again so that no vertex in $B^* - B$ is missed and unmarked. Let U be the set of unmarked vertices in $B^* - B$ missed by M''_{Δ_1+2i}. Let X be the set of big vertices that are endpoints of edges in M''_{Δ_1+2i}. We construct a matching M in $(F - v' - v'' - X) \cap (B_1^*, B_2^*) - M''_{\Delta_1+1} - ... - M''_{\Delta_1+2i-1}$ such that every $v \in U$ is an endpoint of some edge in M. We add M to the final M''_{Δ_1+2i}.

The patching

We now attempt to construct the matchings $M_{\Delta_1+1}, ..., M_{\Delta_1+\Delta_2}$. For $i = 1, ..., \Delta_2$, we obtain M_{Δ_1+i} by augmenting M''_{Δ_1+i} in $H = F - M_{\Delta_1+1} - ... - M_{\Delta_1+i-1}$ with a disjoint matching hitting all unmarked vertices in B missed by M''_{Δ_1+i} as follows:

3. Let U_1 and U_2 be the sets of unmarked vertices missed by M''_{Δ_1+i} in B_1 and B_2, respectively. We attempt to find a perfect matching M in the bipartite subgraph defined by the bipartition (U_1, U_2) and with edge set $E(H) \cap E(U_1, U_2)$.

If we successfully obtain such a matching M, we add its edges to M''_{Δ_1+i} to obtain M_{Δ_1+i}.

If we fail to obtain M, we find the sets $X' \subset U_1$ and $Y' = N^H_{U_2}(X')$ such that $|X'| > |Y'|$ and we set $X'' = U_2 - Y'$ and $Y'' = U_1 - X' = N^H_{U_1}(X'')$. Let $Y = B_2 - X''$ and $X = \{v \in X' : d^H_{B_2}(v) > \frac{1}{2}\Delta - \Delta^{19/20}\}$.

Claim 21 *(X, Y) forms a fail pair in (B_1, B_2).*

Proof: The claim is true if the following three properties hold:

$|Y| < |X| + \Delta^{19/20}$.

Proof: $|Y| \leq |Y'| + 2|M''_{\Delta_1+i}| + m(B_2, \Delta_1 + i) \leq |X'| + \frac{1}{2}\Delta^{9/10} + 2 < |X| + \Delta^{9/10}$. The last inequality follows from $X' - X \subset L$ where $L = \{v \in B : \mathrm{def}(v) > 2\sqrt{\Delta}\}$ is of cardinality $2\sqrt{\Delta}$: since if $v \in B_1 - L$ then $d^H_{B_2}(v) > d^F_{B_2}(v) - i > \frac{1}{2}\Delta - 3\Delta^{9/10} - \frac{1}{2}\Delta^{19/20} < \frac{1}{2}\Delta - \Delta^{19/20}$, and if $v \in B_2 - L$, $d^H_{B_1}(v) > \frac{1}{2}\Delta - \Delta^{19/20}$.

For all v in X: $d_Y(v) > \frac{1}{2}\Delta - \Delta^{19/20}$.

Proof: $d_Y(v) \geq d_Y^H(v) = d_{B_2}^H(v) > \frac{1}{2}\Delta - \Delta^{19/20}$ by definition of X.

$|B_1 - X| > \frac{1}{4}\Delta - \Delta^{19/20}$.

Proof: Since $|X''| > |Y''|$, X'' is not empty. If $v \in X''$ then $d_X^H(v) = 0$. It follows that $|B_1 - X| > d_{B_1}^H(v) > d_{B_1^*}^H(v) - \Delta^{3/5} > \frac{1}{4}\Delta - \Delta^{19/20}$.

□

Chapter 6

The smallest deficiency case

In this chapter we present an algorithm that attempts to color with Δ colors the edges of a multi-Vizing reduction $G^* = (B^* \cup S^*, E^*)$ of a Vizing graph $G = (B \cup S, E)$ such that $\text{def}(B^*) \leq \frac{5}{2}\Delta$. We assume that Δ is large, $|B^* \cup S^*| \leq 6\Delta$, that no trivial subgraph of G^* is overfull and that a modified split partition $(B_1^* \cup S_1^*, B_2^* \cup S_2^*)$ of $B^* \cup S^*$ is given. In case our algorithm fails we will show the existence of and construct a fail pair (X, Y) in (B_1, B_2) (where $B_1 = B \cap B_1^*$ and $B_2 = B \cap B_2^*$).

6.1 Bipartite vs. near-bipartite reductions

In the smallest deficiency case, we once again want to construct about $\frac{1}{2}\Delta$ reducing matchings whose removal leaves a reduction H that is $\Delta(H)$ colorable. One complication is that we can no longer insist that H is bipartite. We illustrate this with the following example. Suppose $|B|$ is even, $|B^* - B| = |S^*| = 0$, $\text{def}(B) = \Delta$, $\text{def}(b_1) = \frac{1}{2}\Delta - 2$, $\text{def}(b_2) = \text{def}(b_3) = \frac{1}{4}\Delta + 1$ where $b_1, b_3 \in B_2$ and $b_2 \in B_1$. Then $c_B = |E(B_1)| - |E(B_2)| = \frac{1}{2}(\text{def}(B_2) - \text{def}(B_1)) = \frac{1}{4}\Delta - 1$. If $M_1, ..., M_k$, for $k = \frac{1}{2}\Delta + o(\Delta)$, were matchings whose removal leaves a bipartite reduction $H = G - M_1 - ... - M_{\Delta_1 + \Delta_2}$ in (B_1, B_2), then both b and b'' must be missed simultaneously by exactly c_B of these matchings. On the other hand, all $d_{B_2}(b'') = \frac{3}{8}\Delta + o(\Delta)$ edges incident to b'' must also be covered by the union of the matchings. We would thus require $k \geq c_B + d_{B_2}(b'') > \frac{5}{8}\Delta + o(\Delta)$, a contradiction. We thus cannot obtain a bipartite reduction with so few matchings.

Our new approach to coloring G^* is to construct disjoint matchings $M_1, ..., M_k$ whose removal leaves either a bipartite reduction or a **near-bipartite reduction** N of G^* with no overfull subgraph of degree $\Delta(N)$. We then color the edges of N with $\Delta(N) = \Delta - k$ colors by applying the edge-coloring algorithm for near-bipartite graphs of Reed[??], and we assign the remaining k colors to the disjoint matchings $M_1, ..., M_k$. In this case, we set $k = \Delta_1 + \Delta_2$ where Δ_1 is the smallest even (odd) integer greater than or equal to $\frac{1}{2}\Delta + \Delta^{3/4}\ln\Delta$ if Δ is even (odd) and Δ_2 is the smallest even integer greater than or equal to $\frac{1}{2}\Delta^{19/20}$.

Let v_1 be the smallest deficiency vertex in B_1 and v_2 be the smallest deficiency vertex in B_2. The near-bipartite reduction N should satisfy the following properties:

A. $E(N) = H + K$ where H is a bipartite graph with edges in $E(B_1^* \cup S_1^*, B_2^* \cup S_2^*)$ and **either** K is a set of edges in $E(B_1)$ incident to v_1 **or** K is a set of edges in $E(B_2)$ incident to v_2,

B. N does not contain a **trivial** overfull subgraph, i.e. B^*, $B^* - v$ for any $v \in B^*$ or $B^* + u$ for any $u \in S^*$ do *not* induce an overfull subgraph in N,

C. $N = (B^* \cup S^*, E(N))$ is **weakly Vizing**, i.e. $d^N_{B^*}(v) \geq \frac{1}{2}\Delta(N)$ for all $v \in B^*$, $d^N_{B^*}(v) \leq \frac{1}{2}\Delta(N)$ for all $v \in S^*$ and $d^N_{B^*}(v) + d^N_{B^*-v}(u) \geq \Delta(N)$ for all $u, v \in B^*$, .

Lemma 23 (near-trivial) *Let $G^* = (B^* \cup S^*, E^*)$ be a multi-Vizing reduction with a modified split partition $(B_1^* \cup S_1^*, B_2^* \cup S_2^*)$, and let N be a subgraph of G^* that satisfies the properties* **A**, **B** *and* **C**. *If N contains an overfull subgraph of maximum degree $\Delta(N)$ then (B_1, B_2) contains a fail pair (X, Y).*

Recall that a fail pair (X, Y) in (B_1, B_2) is a pair of sets such that either:

$$X \subset B_1,\ |B_1 - X| > \frac{1}{4}\Delta - \Delta^{19/20} \text{ and } Y \subset B_2$$

or

$$X \subset B_2,\ |B_2 - X| > \frac{1}{4}\Delta - \Delta^{19/20} \text{ and } Y \subset B_1$$

and, in either case, $|Y| < |X| + \Delta^{19/20}$ and $d_Y(v) > \frac{1}{2}\Delta - \Delta^{19/20}$ for every $v \in X$. If (B_1, B_2) contains a fail pair, we *can* construct a forbidden subgraph in G^* (see 2.5.2). *So, by the near-trivial lemma, our task of Δ edge coloring G^* is reduced to the construction of disjoint matchings whose removal from G^* leaves a reduction N with the properties* **A**, **B** *and* **C**. We accomplish this in two coloring passes which we describe in the remaining of this chapter.

Proof: (of near-trivial lemma 23) Let $N = H + K$ such that $E(H) \subseteq E(B_1^* \cup S_1^*, B_2^* \cup S_2^*)$ and let us assume that K is a subset of edges in $E(B_1)$ incident to v_1 – the proof when K is a subset of edges in $E(B_2)$ incident to v_2 is symmetric. Let F be the non-trivial subgraph of N of maximum degre $\Delta(N)$ (we will slightly abuse our notation by using F to denote the set of vertices of F too). Note that v_1 must belong to F, since otherwise F is bipartite.

We first remark that the set $R = \{v \in F : d_F(v) < \Delta(N) - \sqrt{\Delta(N)}\}$ is smaller than $\sqrt{\Delta(N)}$. Thus any vertex v in the graph H^* induced by $(F - R) \cap (B - b^1)$ has minimum degree greater than $\Delta(N) - 2\sqrt{\Delta(N)} - d_{B^*-B}(v) - \text{mult}(b^1, v) > \Delta(N) - 2\Delta^{3/5} > \frac{1}{2}\Delta - \Delta^{19/20}$. We also note that H^* is bipartite.

Let $X = H^* \cap B_1$ and $Y = H^* \cap B_2$ and let us assume that $|Y| > |X| + \Delta^{19/20}$. Then:

$$\begin{aligned} |E(X, Y)| &> |Y|(\Delta(N) - 2\Delta^{3/5}) \\ &> (|X| + \Delta^{19/20})(\Delta(N) - 2\Delta^{3/5}) \\ &> |X|\Delta(N) > |E(X, Y)| \end{aligned}$$

for $\Delta = \Delta(N) + \Delta_1 + \Delta_2$ large enough. This contradiction and one obtained with a similar argument by assuming that $|X| > |Y| + \Delta^{19/20}$ implies that $||X| - |Y|| < \Delta^{19/20}$. So, if one of $|B_1 - X|$ or $|B_2 - Y|$ is greater than $\frac{1}{4}\Delta - \Delta^{19/20}$, then (X, Y) is a fail pair in (B_1, B_2). In the remainder of this proof, we show that N must contain a trivial overfull subgraph of maximum degree $\Delta(N)$, given the assumption that both $|B_1 - X|$ or $|B_2 - Y|$ are no greater than $\frac{1}{4}\Delta - \Delta^{19/20}$.

Consider the set $C = \{v \in B - F : d_F^N(v) > \sqrt{\Delta(N)}\}$. Since F is overfull, $|C| \leq \sqrt{\Delta(N)}$. Suppose there exists a vertex v in $B - F - C$: v must have at least $\frac{1}{2}\Delta(N) - \Delta^{3/5} > \frac{1}{4}\Delta - \frac{1}{3}\Delta^{19/20}$ neighbors in B, and consequently at least $\frac{1}{4}\Delta - \frac{1}{2}\Delta^{19/20}$ neighbors in $B - F$. Since $v \neq v_1$ and $\text{mult}(v_1, v) \leq 1$, all the neigbors of v in B but at most one (v_1) must be accross the bipartition from v. It follows then that $|B_1 - X| \geq |B_1 - F| \geq \frac{1}{4}\Delta - \frac{1}{2}\Delta^{19/20}$ if $X \subset B_1$

or $|B_2 - X| \geq \frac{1}{4}\Delta - \frac{1}{2}\Delta^{19/20}$ if $X \subset B_2$, contradicting our assumption. So, $B - F - C$ must be empty and $|B - F| \leq \lfloor\sqrt{\Delta(N)}\rfloor$. Suppose now that $|B - F| > 2$ and that $u, v, w \in B - F$. Then, $\Delta(N) > |E(F, B - F)| \geq d_F^N(v) + d_F^N(u) + d_F^N(w) \geq \frac{3}{4}\Delta - 3\Delta^{19/20} > \Delta(N)$, a contradiction. So, there cannot be three vertices in $B - F$ and $|B - F| \leq 2$. We similarly prove $|B^* - F| \leq 2$.

If $B^* - F = \{u, v\}$, then $|E(F + u + v)| = |E(F)| + d_{B^*}^N(u) + d_{B^*-u}^N(v) > \frac{1}{2}\Delta(N)(|F| - 1) + \Delta(N) = \frac{1}{2}\Delta(N)(|F + u + v| - 1)$. So, $F + u + v$ is an overfull subgraph of G^* as well. So, in any case, there exists an overfull subgraph F' of maximum degree $\Delta(N)$ such that $|B^* - F'| \leq 1$.

Suppose now that $|F' \cap S^*| \geq 2$, and let u and v be two vertices of $F' \cap S^*$. Then $|E(F')| = |E(F' - u - v)| + d_F^N(u) + d_F^N(v) \leq \frac{1}{2}\Delta(N)(|F'| - 2) - (d_F^N(u) + d_F^N(v)) + (d_F^N(u) + d_F^N(v)) = \frac{1}{2}\Delta(N)(|F'| - 2)$ contradicting the fact that F' is overfull. So, $|F' \cap S^*| \leq 1$.

Finally, we show that if $|B^* - F'| = 1$ and $|S^* \cap F'| = 1$, then B^* itself must be overfull. This would prove our claim that G^* must contain a trivial overfull subgraph. Suppose that $F' = B^* - v + u$ for some $v \in B^*$ and $u \in S^*$. Then $|E(F')| = |E(F' - u)| + d_F^N(u) \leq |E(B^*)| - d_{B^*}^N(v) + d_{B^*}^N(u) \leq |E(B^*)|$ since $d_{B^*}^N(v) \geq d_{B^*}^N(u)$. Since $|E(F')| > \frac{1}{2}\Delta(N)(|F'| - 1) = \frac{1}{2}\Delta(N)(|B^*| - 1)$, B^* must be overfull.

□

6.2 The first coloring pass

In the first pass, we attempt to construct the matchings $M_1, ..., M_{\Delta_1}$ containing most edges in $E(B_1^* \cup S_1^*)$ and $E(B_2^* \cup S_2^*)$ such that $F = G^* - \cup_{i=1}^{\Delta_1} M_i$ is a reduction of G^*. As in the previous, higher deficiency, cases, we obtain these matchings by patching the unmarked big vertices missed by a set of initial matchings. Our patching technique is essentially the same as in the small deficiency case; so, we focus our attention to the marking. The issues that drive our marking choices include the same ones as in the previous cases. If $|B^*|$ is odd (even), an odd (even) number of big vertices must be missed by the big edges of every matching. Also, the patching step requires that we keep the degree of every vertex high, so we must insist that every vertex with large deficiency is missed by many matchings. In the smallest deficiency case, the marking must satisfy a few additional conditions if the final near-bipartite reduction is to satisfy the properties **A**, **B** and **C**. To make matters more complicated, we must be very precise how we define the marking, because very little deficiency is available. For this purpose, we find it useful to first assign **targets** to every $v \in B^*$, i.e. the number of matchings among $M_1, ..., M_{\Delta_1}$ whose big edges miss v. Because of the different "types" of deficiency we specify different types of targets. We denote by $t_r(v)$ the number of matchings missing v and by $t_{s_1}(v)$, $t_{s_2}(v)$ and $t_{s_3}(v)$, the number of matchings containing the edge (s_1, v), (s_2, v) and (s_2, v). We also use $t(v)$ to denote $t_r(v) + \sum_{u \in S^*} t_u(v)$.

We find it convenient to describe the target assignments separately for the cases when $|B^*|$ is odd or even. Once we have a satisfactory target asssignment, we define a marking that is proper over an intial set of Δ_1 matchings such that $m_u^{\Delta_1}(v) = \sum_{i=1}^{\Delta_1} m(v,i) = t_u(v)$ for every $v \in B^*$, $u = r$ and $u \in S^*$.

6.2.1 The targets for odd $|B^*|$

The target conditions

If $|B^*|$ is odd, the big edges of every matching $M_1, ..., M_{\Delta_1}$ must miss an odd number of big vertices. In addition, we insist that every edge in $E(B_1^*, S_1^*)$ and in $E(B_2^*, S_2^*)$ must belong to some matching. The target assignments must then satisfy

$$t(B^*) = \sum_{v \in B^*} t(v) \geq \Delta_1 \tag{6.1}$$

and

$$t_{s_1}(B_2^*) = \mathrm{def}_{s_1}(B_2^*),\ t_{s_2}(B_1^*) = \mathrm{def}_{s_2}(B_1^*) \text{ and } t_{s_3}(B_2^*) = \mathrm{def}_{s_3}(B_2^*) \tag{6.2}$$

We make sure that B^* does not induce an overfull subgraph in F with

$$\mathrm{def}(B^*) - t(B^*) \geq \Delta - \Delta_1 \tag{6.3}$$

In G^*, by the properties of a modified split partition, there are c_B more edges in B_1^* than in B_2^* (where $\frac{1}{4}\Delta < c_B \leq \frac{1}{2}\Delta$). Suppose k_1 is the integer defined by $t(B_1^*) - t(B_2^*) = \Delta_1 - 2(c_B - k_1)$. Then, k_1 more big edges in B_1^* than in B_2^* will not be covered by $\cup_{i=1}^{\Delta_1} M_i$. Since a subset of these edges eventually forms K, the edges of the final near-bipartite reduction N whose removal from N leaves a bipartite graph, we must choose these k_1 edges to be incident to v_1 and we insist that

$$t(B_1^*) - t(B_2^*) = \Delta_1 - 2(c_B - k_1) \text{ for } \max\{0, \frac{1}{8}\Delta - \frac{1}{2}c_S - 2\delta\} \leq k_1 \leq \frac{1}{4}\Delta + \delta \tag{6.4}$$

Recall that $c_S = \frac{1}{2}(d(S_2) - d(S_1)) \leq \frac{1}{2}d(s_1)$. The curious lower bound on k_1 is a technical condition that we will use in the reject coloring pass. If our target assignments are such that $k_1 = 0$ in 6.4, we will construct a bipartite reduction H of G^* in the two coloring passes. If, however, our target assignments are such that $k_1 > 0$ then we may construct a near-bipartite reduction instead. In that case, for technical reasons that help simplify our analysis and to ultimately satisfy property **C**, we require that big vertices stay big and small vertices stay small. More precisely, we require that $d_{B^*}^F(v) \geq \frac{1}{2}(\Delta - \Delta_1)$ for every $v \in B^*$ and $d_{B^*}^F(u) \leq \frac{1}{2}(\Delta - \Delta_1)$ for every $u \in S^*$. We will obtain this if the target assignments satisfy

$$\text{def}(v) - t(v) \leq \frac{1}{2}(\Delta - \Delta_1) \text{ for } v \in B^* \tag{6.5}$$

$$\text{def}_u(B^*) - t_u(B^*) \leq \frac{1}{2}(\Delta - \Delta_1) \text{ for } u \in S^* \tag{6.6}$$

and $\text{def}(v) - t(v) + \text{def}(u) - t(u) + \mu^F(u,v) \leq \Delta - \Delta_1$ for all $u, v \in B^*$, where $\mu^F(u,v)$ is the multiplicity of edge (u,v) in F. In almost all cases, our target assignments will satisfy the stronger condition

$$\text{def}(v) - t(v) + \text{def}(u) - t(u) + \mu(u,v) \leq \Delta - \Delta_1 \text{ for } u, v \in B^* \tag{6.7}$$

where $\mu(u,v)$ is the multiplicity of edge (u,v) in G^*. In some cases, when all the deficiency is concentrated in only two vertices $v = b_1$ and $u = b_2$, we are not be able to insure 6.7 for the pair b_1, b_2. In this special case, $t(B^* - b_1 - b_2) = \text{def}(B^* - b_1 - b_2)$ and $\mu(b_1, b_2) = 1$, and we will insist that one of $M_1, ..., M_{\Delta_1}$ contains (b_1, b_2) (which we take care of in the marking step).

Finally, we require a number of technical conditions to be satisfied by the targets to help us define the marking and construct the matchings $M_1, ..., M_{\Delta_1}$. First, we find it useful to limit the number of marks to 3 per matching, and we thus require

$$t(B^*) < 3\Delta_1 \tag{6.8}$$

and also

$$2(t(u) + t(v)) \leq \Delta_1 + t(B^*) \text{ for } u, v \in B^* \tag{6.9}$$
$$2(t_u(B^*) + t_v(B^*)) \leq \Delta_1 + t(B^*) \text{ for } u, v \in S^* \tag{6.10}$$
$$2(t(v) + t_u(B^* - v)) \leq \Delta_1 + t(B^*) \text{ for } v \in B^*, u \in S^* \tag{6.11}$$

We also insist that most vertices v are not marked much more than $\frac{1}{2}\mathrm{def}(v) + \frac{1}{2}\delta$ times.

$$\sum_{v \in B^*} (t(v) - (\frac{1}{2}\mathrm{def}(v) + \frac{1}{2}\delta)) < \delta \tag{6.12}$$

The target assignments

Initially, let $t_r(v) = 0$ and $t_u(v) = 0$ for every $v \in B^*$ and $u \in S^*$. In the following three steps, we define an initial target assignment that satisfies most of the target conditions:

1.1 For every $(u, v) \in (B_1^*, S_1^*) \cup (B_2^*, S_2^*)$, we set $t_u(v) = \mathrm{def}_u(v) = \mu(u, v)$.

We call a vertex $u \in S^*$ **bad** if $\mathrm{def}_u(B^*) - t_u(B^*) > \frac{1}{2}(\Delta - \Delta_1)$.

1.2 For every bad $u \in S^*$, we repeat the following until $\mathrm{def}_u(B^*) - t_u(B^*) = \frac{1}{2}(\Delta - \Delta_1)$:

we choose $v \in B^*$ with positive $\mathrm{def}_u(v) - t_u(v)$ (note that u and v must be on opposite sides of the bipartition) and we add 1 to $t_u(v)$; if we have a choice, we always choose v with largest $\mathrm{def}(v) - t(v)$.

It is easy to check that $|t(B_2^*) - t(B_1^*) - c_S| < \frac{10}{8}\delta$ after step **1.2**. We call a vertex $v \in B^*$ **bad** if $\mathrm{def}(v) - t(v) > \frac{1}{2}(\Delta - \Delta_1)$.

1.3 For every bad v in B^*, we set $t_r(v) = \mathrm{def}(v) - \frac{1}{2}(\Delta - \Delta_1) - t(v)$ if v is bad and $t_r(v) = 0$ otherwise.

We explicitely made sure that target conditions 6.2, 6.5 and 6.6 are satisfied by these initial assignments. Since $t(v) < \frac{1}{2}\mathrm{def}(v) + \frac{1}{2}\delta$ for all $v \in B^*$ and $t_u(B^*) \leq \frac{1}{2}\mathrm{def}_u(B^*) + \frac{1}{2}\delta$ for all $u \in S^*$, conditions 6.9, 6.10, 6.11 and 6.12 are trivially satisfied as well. We note, furthermore, that $t_u(B^*) \leq \frac{1}{2}\mathrm{def}_u(B^*) + \frac{1}{8}\delta$ for all non-bad $u \in S^*$, and $t(v) \leq \frac{1}{2}\mathrm{def}(v)$ for all $v \in B^*$ with deficiency greater than $\Delta - \Delta_1$. So, $t(B^*) < \frac{1}{2}\mathrm{def}_{S^*}(B^*) + \frac{3}{2}\delta + \frac{1}{2}\mathrm{def}_r(B^*) + \frac{5}{2}\delta \leq \frac{1}{2}\mathrm{def}(B^*) + 4\delta \leq \frac{5}{4}\Delta + 4\delta$ which is much less than $3\Delta_1$. Thus, condition 6.8 is satisfied.

We show next that 6.3 holds. If there are more than two bad (big or small) vertices then $\mathrm{def}(B^*) - t(B^*) > \frac{3}{2}(\Delta - \Delta_1 - \Delta^{9/10}) > \Delta - \Delta_1$. If there are less than two bad vertices then $t(B^*) < \frac{1}{2}\mathrm{def}(B^*) + \frac{7}{8}\delta$, implying $\mathrm{def}(B^*) - t(B^*) \geq \frac{1}{2}\mathrm{def}(B^*) - \frac{7}{8}\delta > \Delta - \Delta_1$. Finally, if exactly two vertices u and v are bad, then $\mathrm{def}(B^*) - t(B^*) \geq \mathrm{def}(u) - t(u) + \mathrm{def}(v) - t(v) \geq \Delta - \Delta_1$ if $u, v \in B^*$, and $\mathrm{def}(B^*) - t(B^*) \geq \mathrm{def}_u(B^*) - t_u(B^*) + \mathrm{def}_v(B^*) - t_v(B^*) \geq \Delta - \Delta_1$ if $u, v \in S^*$. A similar argument shows that 6.3 holds for $u \in S^*$ and $v \in B^*$, except if some of v's remaining deficiency is induced by u. In that case, however, by our choices of vertices in step **1.2**, $t_u(B^*) \leq \frac{1}{2}\mathrm{def}_u(B^*) + \frac{1}{8}\delta + \sqrt{\Delta}$. Since $t(v) \leq \frac{1}{2}\mathrm{def}(v) + \frac{1}{2}\delta$, it follows that $t(B^*) \leq \frac{1}{2}\mathrm{def}(B^*) + \frac{7}{8}\delta + \sqrt{\Delta}$, implying $\mathrm{def}(B^*) - t(B^*) > \Delta - \Delta_1$.

It is clear that 6.7 holds if $u, v \in B^* - B$. If $u \in B^* - B$, $v \in B$ and $\mu(u,v) \geq 2$ then both $\mathrm{def}(u) - t(u)$ and $\mathrm{def}(v) - t(v)$ are greater than $\frac{1}{2}(\Delta - \Delta_1) - \mu(u,v) > \frac{1}{4}\Delta - \Delta^{9/10}$. By claim 16, it follows that $\mu(u,v) \leq 3$ and $\mathrm{def}(v) < \frac{3}{8}\Delta + \Delta^{9/10}$. Thus, we see that $t(v) < \frac{1}{8}\Delta + 2\Delta^{9/10}$, implying $t(v) < \frac{1}{2}\mathrm{def}(v) - \frac{1}{16}\Delta + 2\Delta^{9/10}$ and also $t(B^*) < \frac{1}{2}\mathrm{def}(B^*) - \frac{1}{16}\Delta + 3\Delta^{9/10}$. So, we can insure 6.7, while maintaining the validity of the conditions we just checked, as follows:

1.4 We add $\mu(u,v)$ to $t_r(v)$ for every $v \in B$ for which there exists some $u \in B^*$ such that 6.7 does not hold.

Note that the total we add to $t(B^*)$ is less than 30 since there can be no more than 10 vertices of deficiency greater than $\frac{1}{4}\Delta - \Delta^{9/10}$. If $\mu(u,v) = 1$, we do step **1.4** as well, unless 6.3 fails to hold as a result (all previously checked target conditions remain satisfied). If 6.3 would fail, we do nothing as this is our special case.

It remains for us to modify, if necessary, our target assignments to satisfy conditions 6.1 and 6.4. Before starting that, we compute the difference $t(B_1^*) - t(B_2^*)$. Let B_1^b and B_2^b be the sets of bad big vertices in B_1^* and B_2^*, respectively, and let us assume that $|B_1^b| = |B_2^b|$. Then $t(B_2^*) - t(B_1^*)$ is within 30 of $\sum_{v\in B_1^b}(\mathrm{def}(v) - \frac{1}{2}(\Delta - \Delta_1)) - \sum_{v\in B_2^b}(\mathrm{def}(v) - \frac{1}{2}(\Delta - \Delta_1)) + t(B_1^* - B_1^b) - t(B_2^* - B_2^b) = \mathrm{def}(B_1^*) - \mathrm{def}(B_2^*) - (\mathrm{def}(B_1^* - B_1^b) - \mathrm{def}(B_2^* - B_2^b)) + t(B_1^* - B_1^b) - t(B_2^* - B_2^b)$, and since $\mathrm{def}(B_1^*) - \mathrm{def}(B_2^*) = \Delta_1 - 2c_B + (\Delta - \Delta_1)$ and $t(B_1^* - B_1^b) - t(B_2^* - B_2^b)$ is within $30\sqrt{\Delta}$ of $t_{S^*}(B_1^*) - t_{S^*}(B_2^*)$, it follows that $t(B_1^*) - t(B_2^*) = \Delta_1 - 2(c_B - k_1)$ where k_1 is within $\frac{3}{2}\delta$ of $\frac{1}{2}((\Delta - \Delta_1) - (\mathrm{def}(B_1^* - B_1^b) - \mathrm{def}(B_2^* - B_2^b)) - c_S)$ and thus $\frac{1}{8}\Delta - \frac{1}{2}c_S - 2\delta < k_1 < \frac{1}{4}\Delta - c_S + \delta$. Using a similar argument, we obtain the same result if $B_1^b| = |B_2^b| + 1$.

If $t(B^*)$ and Δ_1 have the same parity, then k_1 is integer, as desired; otherwise,

1.5 We add 1 to $t_u(v)$ for some $v \in B^*$ with positive $\mathrm{def}_u(v) - t_u(v)$.

Using simple parity arguments, we can see that all previously checked target conditions are satisfied. If $t(B^*) \geq \Delta_1$ but $t(B_1^*) - t(B_2^*) < \Delta_1 - 2c_B$, we add more to $t(B_1^*)$ until we obtain $t(B_1^*) - t(B_2^*) = \Delta_1 - 2c_B$ by repeating the following $\frac{1}{2}(\Delta_1 - 2c_B - (t(B_1^*) - t(B_2^*)))$ times:

1.6 We add 1 to $t_{u_1}(v_1)$ and $t_{u_2}(v_2)$ for some $v_1, v_2 \in B_1^*$ with positive $\mathrm{def}_{u_1}(v_1) > t_{u_1}(v_1)$ and $\mathrm{def}_{u_2}(v_2) > t_{u_2}(v_2)$ where $u_1 = r$ or $u_1 = s_1$ and $u_2 = r$ or $u_2 = s_3$.

Note that we maintain the validity of target conditions 6.9 - 6.11 thanks to our choices of vertices in every iteration and because $t(B_1^*) - t(B_2^*) \leq \Delta_1 - 2c_B$. Since we repeat **1.5** at most δ times, it follows that 6.12 is satisfied.

If $t(B^*) < \Delta_1$, we also must add more to $t(B^*)$ until we obtain $t(B^*) = \Delta_1$. Note that if $t(B^*) = \Delta_1$, all target conditions, except possibly 6.4 and 6.12, are trivially satisfied. We repeat the following until $t(B^*) = \Delta_1$:

1.7 We add 1 to $t_u(v)$ for some v in B^* with $\mathrm{def}_u(v) > t_u(v)$ for $u = r$ or $u \in S^*$ and with $\frac{1}{2}\mathrm{def}(v) + \frac{1}{2}\delta - t(v)$; we choose $v \in B_1^*$ if $t(B_1^*) - t(B_2^*) < \frac{1}{2}\Delta - c_B (= \frac{1}{2}\mathrm{def}(B_1^*) + \frac{1}{2}\delta - \frac{1}{2}\mathrm{def}(B_2^*) - \frac{1}{2}\delta$, otherwise we choose $v \in B_2^*$.

If $t(B_1^*) - t(B_2^*) < \Delta_1 - 2c_B$ after step **1.7**, we continue exactlt as in **1.6**. Otherwise, all target conditions hold.

6.2.2 The targets for even $|B^*|$

The target conditions

If $|B^*|$ is even, the number of big vertices missed by the big edges of every matching must be even. It follows that $t(B^*)$ must be even too. In other words, if k_1 satisfies $t(B_1^*) - t(B_2^*) = 2(c_B - k_1)$ then k_1 must be integer. In addition, if $k_1 > 0$ then k_1 more big edges in B_1^* than in B_2^* will not be covered by $\cup_{i=1}^{\Delta_1} M_i$, and if $k_1 < 0$ then $|k_1|$ more big edges in B_2^* will not be covered by $\cup_{i=1}^{\Delta_1} M_i$. Since a subset of these edges eventually forms K, the edges of the final near-bipartite reduction N whose removal from N leaves a bipartite graph, we must choose these $|k_1|$ edges to be incident to v_1 or v_2, depending on whether $k_1 > 0$ or $k_1 < 0$. For this purpose we insist that

$$t(B_1^*) - t(B_2^*) = 2(c_B - k_1) \text{ for } -\frac{1}{8}\Delta - \delta \leq k_1 \leq \frac{1}{8}\Delta + \delta \qquad (6.13)$$

The following target conditions are necessary and sufficient to insure that $B^* - v$, for every $v \in B^*$, and $B^* + u$, for every $u \in S^*$, do not induce an overfull subgraph in F

$$2(\text{def}(v) - t(v)) \leq \text{def}(B^*) - t(B^*) \qquad (6.14)$$
$$2(\text{def}_u(B^*) - t_u(B^*)) \leq \text{def}(B^*) - t(B^*) \qquad (6.15)$$

To help our analysis and to ultimately satisfy property **C**, we require that big vertices stay big and small vertices stay small, and so properties 6.5, 6.6 and 6.7 should be satisfied by the targets. We also insist on 6.8 to limit the number of marks per matching, 6.2 to include all small edges in $E(B_1^*, S_1^*) \cup E(B_2^*, S_2^*)$ in the first Δ_1 matchings and on property 6.12 so most vertices v are not marked much more than $\frac{1}{2}\text{def}(v) + \frac{1}{2}\delta$ times. Finally, in order to put an even number of marks per matching in the marking step, we need that the targets satisfy:

$$2t(v) \leq t(B^*) \text{ for all } v \in B^* \qquad (6.16)$$
$$2t_u(B^*) \leq t(B^*) \text{ for all } u, v \in S^* \qquad (6.17)$$

The target assignments

We initially assign targets as we did in steps **1.1**, **1.2**, **1.3** and **1.4** of the odd case. This assignment satisfies condition 6.2 requiring that all small edges in $E(B_1^*, S_1^*) \cup E(B_2^*, S_2^*)$ belong to the first Δ_1 matchings, and also conditions 6.5, 6.6 and 6.7 insuring that smalls stay small and bigs stay big. The total number of targets, $t(B^*)$, is much less than $3\Delta_1$ (required in the marking step) so 6.8 holds as well. Finally, we observe that our assignments satisfy $t(v) \leq \frac{1}{2}\mathrm{def}(v) + \frac{1}{2}\delta + 3$ for all $v \in B^*$, and, as in the odd $|B^*|$ case, $t_{S^*}(B_2^*) - t_{S^*}(B_1^*)$ is within $\frac{10}{8}\delta$ of c.

Let B_1^b and B_2^b be the number of bad vertices in B_1^* and B_2^*, respectively. If $|B_1^b| = |B_2^b|$, the difference $t(B_2^*) - t(B_1^*)$ is within 30 of $\mathrm{def}(B_2^*) - \mathrm{def}(B_1^*) - (\mathrm{def}(B_2^* - B_2^b) - \mathrm{def}(B_1^* - B_1^b)) + (t(B_2^* - B_2^b) - t(B_1^* - B_1^b))$. Since $t(B_2^* - B_2^b) - t(B_1^* - B_1^b)$ is within $\frac{10}{8}\delta + 30\sqrt{\Delta}$ of c_S, it follows that $t(B_2^*) - t(B_1^*) = 2(c_b - k_1)$ where $|k_1| < \frac{1}{8}\Delta + \delta - 1$. The same is true if $|B_1^b| + 1 = |B_2^b|$. Thus, 6.13 is satisfied, unless k_1 is not integer, in which case we just do

1.5 If $t(B^*)$ is odd (i.e. k_1 is not integer) but all other properties hold, we just add 1 to $t_u(v)$ for some $v \in B^*$ with $\mathrm{def}_u(v) > t_u(v)$ for $u = r$ or $u = s, s', s''$.

Since $\mathrm{def}(B^*)$ is even, there must exist such a vertex v, and by parity arguments, all checked conditions are still valid.

We now modify the targets, if necessary, so the conditions 6.14, 6.15, 6.16 and 6.17 hold:

1.6 We recursively add 1 to $t_u(v)$ for some $v \in B^*$ and $u = r$ or $u \in S^*$ such that $\mathrm{def}_u(v) > t_u(v)$ while one of the following holds:

- **(i)** there is some $v_1 \in B^*$ with $q_1 = 2t(v_1) - t(B^*) > 0$ (more than half of the used targets (defined as $t(B^*)$) is on v_1),
- **(ii)** there is $v_2 \in B^*$ with $q_2 = 2(\mathrm{def}(v_2) - t(v_2)) - (\mathrm{def}(B^*) - t(B^*)) > 0$, i.e more than half of the available deficiency (defined as $\mathrm{def}(B^*) - t(B^*)$) is on v_2),
- **(iii)** there is some $u_1 \in S^*$ with $q_3 = 2t_{u_1}(B^*) - t(B^*) > 0$ (more than half of the used targets is induced by u_1),
- **(iv)** there is $u_2 \in S^*$ with $q_4 = 2(\mathrm{def}_{u_2}(B^*) - t_{u_2}(B^*)) - (\mathrm{def}(B^*) - t(B^*)) > 0$ (more than half of the available deficiency is induced by u_2).

We first note that more than one of the above inequalities may hold. Actually, all of them may hold. There can be, however, only one vertex satisfying one particular condition. We pick u and v so that $\mathrm{def}_u(v) > t_u(v)$ and all of the following are satisfied:

if **(i)** holds, $v \in B^* - v_1$,
if **(ii)** holds, $v = v_2$,
if **(iii)** holds, $u = r$ or $u \in S^* - u_1$,
if **(iv)** holds, $u = u_2$.

Finally, if we have a choice of vertex v to mark (i.e. if **(ii)** does not hold), we choose $v \in B_1^*$ if $t(B_2^*) - t(B_1^*) > c_B$ or $v \in B_2^*$ if $t(B_2^*) - t(B_1^*) \le c_B$.

The number of iterations of **1.6**, i.e. the total we add to $t(B^*)$, is $q = \max\{q_1, q_2, q_3, q_4\}$. If $q = q_1$, then the total $t(B^*)$ after **1.6** is at most $2t(v_1)$ and since $t(v_1) < \frac{1}{2}\mathrm{def}(v_1)+\frac{1}{2}\delta+4$, it follows that $t(v) \le \frac{1}{2}\mathrm{def}(v)+\frac{1}{2}\delta+4$, for all $v \in B^*$ implying target condition 6.12; a similar argument works when $q = q_3$. If $q = q_2$, then either there is a bad vertex in $(B^*-v_2)\cup S^*$ and $q_2 \le 4$, or $t(B^*-v_2) < \frac{1}{2}\mathrm{def}(B^*-v_2)+\frac{1}{2}\delta$ and $q_2 = \mathrm{def}(v_2)-t(v_2)-\frac{1}{2}\mathrm{def}(B^*-v_2)+\frac{1}{2}\delta \le \frac{1}{2}\mathrm{def}(v_2) - t(v_2) + \frac{1}{2}\delta$. In either case $t(v_2) \le \frac{1}{2}\mathrm{def}(v_2) + \frac{1}{2}\delta$, implying 6.12. A similar argument works when $q = q_4$.

6.2.3 The initial coloring

We construct disjoint matchings $M_1', ..., M_{\Delta_1}'$ such that $\cup_{i=1}^{\Delta_1} M_i' = E(B_1^*) \cup E(B_2^*) - K_1$, where K_1 is a set big edges chosen as follows:

2.1 If $k_1 > 0$ (where k_1 is as defined in 6.4 for the odd $|B^*|$ case or in 6.15 for the even $|B^*|$ case) then we choose some k_1 edges in $E(B_1)$ incident to v_1, the smallest deficiency vertex in B_1; if $k_1 < 0$, we choose some $|k_1|$ edges in $E(B_2)$ incident to v_2, the smallest deficiency vertex in B_2.

Because $d_{B^*}(v_1) > 3$, it follows that $d_B(v_1) > \Delta - \Delta^{3/5} - 3$. If $k_1 > 0$, then $k_1 < \frac{1}{4}\Delta + 2\delta < \frac{1}{2}d_{B_1}(v_1) + \Delta^{9/10} < d_{B_1}(v_1)$. Similarly, $|k_1| < d_{B_2}(v_2)$ if $k_1 < 0$.

Because $\Delta(B_1^*), \Delta(B_2^*) < \frac{1}{2}\Delta + \frac{1}{8}\delta < \Delta_1 - \sqrt{\Delta}$, and because the edge multiplicity in B_1^* and B_2^* is at most $\sqrt{\Delta}$, we can apply Fournier's edge coloring algorithm for multigraphs to Δ_1 color B_1^* and B_2^*. We can thus construct an initial coloring as follows:

2.2 We color the edges of $E(B_1^*) \cup E(B_2^*) - K_1$ with Δ_1 colors using Fournier's algorithm to obtain matchings $M_1^1, ..., M_{\Delta_1}^1$ such that $\cup_{i=1}^{\Delta_1} M_i^1 = E(B_1^*) - K_1$ and matchings $M_1^2, ..., M_{\Delta_1}^2$ such that $\cup_{i=1}^{\Delta_1} M_i^2 = E(B_2^*) - K_1$. By applying the balancing procedure from 3.2.2, we insure that $M_1^1, ..., M_{\Delta_1}^1$ are balanced in B_1^* and $M_1^2, ..., M_{\Delta_1}^2$ are balanced in B_2^*.

2.3 We set $M_i' = M_i^1 \cup M_i^2$ for $i = 1, ..., \Delta_1$.

Note that $0 \leq |M_j^1| - |M_i^1| \leq 1$ and $0 \leq |M_j^2| - |M_i^2| \leq 1$ for $1 \leq i < j \leq \Delta_1$, and $0 \leq |M_i^1| - |M_i^2| \leq 1$ for $1 \leq i \leq \Delta_1$. Actually, $|M_i^1| - |M_i^2| = 1$ for exactly $c_B - k_1$ indices i.

6.2.4 The marking

Using the targets, and by reordering and modifying the initial matchings, we define a proper marking over modified disjoint matchings $M_1', ..., M_{\Delta_1}'$ such that $\cup_{i=1}^{\Delta_1} M_i' = (E(B_1^*) - E(R_1)) \cup (E(B_2^*) - E(R_2)) - K_1$ where:

(i) $m_r^{\Delta_1}(v) = t_r(v)$ and $m_u^{\Delta_1}(v) = t_u(v)$ for every $v \in B^*$ and $u \in S^*$.

(ii) $\delta < n(B_1^*, i) = n(B_2^*, i) < 7\delta$ for every $i = 1, ..., \Delta_1$,

(iii) R_1 and R_2 are reject subgraphs of maximum degree less than $\Delta^{4/5}$ and $|E(R_1)| = |E(R_2)| < 2\Delta$.

We recall that $n(B_1^*, i)$ and $n(B_2^*, i)$ are the numbers of vertices v in B_1^* and B_2^*, respectively, missed by M_i' such that $m(v, i) = 0$. These are exactly the vertices we must patch later. We also recall that a marking is proper over initial matchings $M_1', ..., M_{\Delta_1}'$ if the big edges of every M_i' miss every big vertex v such that $m(v, i) = 1$.

We start the marking procedure with $m_u(v, i) = 0$ for all $v \in B^*$, $u = r$ and $u \in S^*$, and all $i = 1, ..., \Delta_1$. We also call all M_i' **unused**. Then we repeat the following for $i = 1, ..., \Delta_1$:

3.1 We mark a few of the big vertices, by setting $m_r(v, i) = 1$ or $m_u(v, i) = 1$ for a few $v \in B^*$ and $u \in S^*$.

3.2 We choose some unused M'_j and switch it with M'_i. We then modify (the new) M'_i so it misses all vertices v such that $m(v,i)=1$. Finally, we add the small edges induced by the marking to M'_i and call M'_i **used**. Note that $M'_1,...,M'_i$ are used after this step, while $M'_{i+1},...,M'_{\Delta_1}$ are unused.

3.3 We delete up to four big edges of M'_i on one side of the bipartition to insure that $n(B^*_1,i)=n(B^*_2,i)$.

We now give the details of each step.

Step 3.1: choosing the marks

We describe first how we choose what vertices to mark in iteration i. For that purpose, we introduce some notation. We use $m^i_u(v)$ to denote $\sum_{j=1}^{i} m_u(v,j)$ and $m^i(v)$ to denote $m^i_r(v)+\sum_{u\in S^*} m^i_u(v)$, and we say that $v\in B^*\cup S^*$ has, in iteration i, y available targets if either $v\in B^*$ and $y=t(v)-m^{i-1}(v)$ or $v\in S^*$ and $y=t_v(B^*)-m^{i-1}_v(B^*)$, respectively.

If $|B^*|$ is odd, we set $x=\frac{1}{2}(t(B^*)-\Delta_1)$ which is less than Δ_1 since $t(B^*)<3\Delta_1$ by target condition 6.1. Then, in iteration i, we mark the vertices as follows:

3.1.1 If $i\leq x$, we set $m_{u_1}(v_1,i)=m_{u_2}(v_2,i)=m_{u_3}(v_3,i)=1$ for some different vertices $v_1,v_2,v_3\in B^*$ and $u_j=r$ or $u_j\in S^*$ such that $t_{u_j}(v_j)>m^{i-1}_{u_j}(v_j)$ for $j=1,...,3$ and no two u_k and u_l $(1\leq l<k\leq 3)$ are the same small vertex. We insist that the two vertices with largest numbers of available targets are among v_1,v_2,u_1,u_2.

3.1.2 If $i>x$, we set $m_r(v,i)=1$ for some $v\in B^*$ with $t_r(v)>m^{i-1}_r(v)$ or we set $m_u(v,i)=1$ for some $v\in B^*$ and $u\in S^*$ with $t_u(v)>m^{i-1}_u(v)$. In the special case when target condition 6.8 holds, we choose, in iteration $i=x+1$, $v\in B^*-b_1-b_2$, and we also temporarily mark b_1 and b_2 (because we will add the edge (b_1,b_2) to M_i in teh marking step).

Before every iteration $i\leq x$, every vertex $v\in B^*\cup S^*$ has at most $\Delta_1-(i-1)$ available targets, and at most 2 have exactly $\Delta_1-(i-1)$ available targets (since $t(B^*)<3\Delta_1$, by 6.9). If only two vertices v_1 and v_2 with positive available targets remain just before iteration $i\leq x$, then our choice of vertices to mark would contradict conditions 6.9, 6.10 or 6.11. When

$i > x$, $t(B^*) - m^{i-1}(B^*) = \Delta_1 - (i-1)$, implying that we always *can* choose the vertices to mark as required.

If $|B^*|$ is even, we set $x = \max\{0, \frac{1}{2}(t(B^*) - 2\Delta_1)\}$ and $y = \min\{\Delta_1, \frac{1}{2}t(B^*)\}$. Note that either $x = 0$ or $y = 0$ but not bth. Since $t(B^*)$ is much less tahn $3\Delta_1$ (by target condition 6.1), it follows that $x < \frac{1}{4}\Delta + \frac{1}{2}\delta$. Then, in iteration i, we mark the vertices as follows:

3.1.3 If $i \leq x$, we set $m_{u_1}(v_1, i) = m_{u_2}(v_2, i) = m_{u_3}(v_3, i) = m_{u_4}(v_4, i) = 1$ for some different vertices $v_1, v_2, v_3, v_4 \in B^*$ and $u_j = r$ or $u_j =\in S^*$ such that $t_{u_j}(v_j) > m_{u_j}^{i-1}(v_j)$ for $j = 1, ..., 4$ and no two u_k and u_l $(1 \leq l < k \leq 4)$ are the same small vertex. We insist that the two vertices with largest numbers of available targets are among v_1, v_2, u_1, u_2.

3.1.4 If $x < i \leq y$, we set $m_{u_1}(v_1, i) = m_{u_2}(v_2, i) = 1$ for some different vertices $v_1, v_2 \in B^*$ and $u_j = r$ or $u_j \in S^*$ such that $t_{u_j}(v_j) > m_{u_j}^{i-1}(v_j)$ for $j = 1, 2$ and u_1 and u_2 are *not* the same small vertex. We insist that the two vertices with largest numbers of available targets are among v_1, v_2, u_1, u_2.

3.1.5 If $i > y$, we mark no vertex.

It is easy to check that our target conditions insure the feasibility of this marking.

Step 3.2: constructing the matching

We start by calling all matchings $M'_1, ..., M'_{\Delta_1}$ unused. Then, in every iteration i, we associate with every v such that $m(v, i) = 1$) a matching M'_{i_v} missing v such that $m(v, i_v) = 0$, if there is such a matching. From target condition 6.10, and since v is missed by at least $\frac{1}{2}\text{def}(v) + \frac{7}{8}\delta$ initial matchings, we see that such a matching M'_{i_v} must exist for every marked v in all but, at most, 2δ iterations i. We then choose some unused matching M'_j, switch it with M'_i and, then, we modify (the new) M'_i so it misses every marked vertex v. We then add the small edges induced by the marking and the edge (b_1, b_2), in the special case, to M'_i and call it used: so, the used matchings are $M'_1, ..., M'_i$ after iteration i.

For every vertex v that is hit by M'_i (with an edge (u, v)), we use one of the following procedures:

first edge reject: We simply reject (v, u), as shown in figure 6.1.

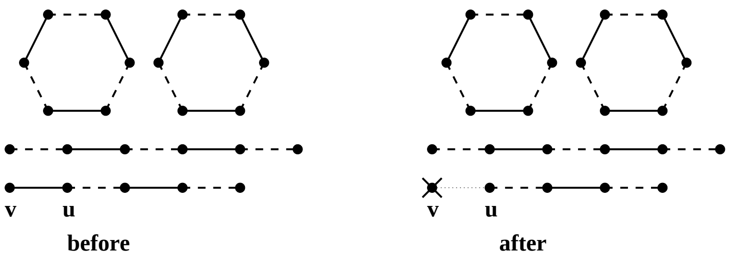

Figure 6.1: M'_i (full) and M'_{i_v} (dashed) before and after, with v marked in M'_i, the first edge reject procedure

any edge reject If M'_{i_v} misses u and if either M'_{i_v} is unused or M'_{i_v} is used and $m(u, i_v) = 0$, we switch the color of (v, u) from M'_i to M'_{i_v} and we reject an edge (x, y) in M'_{i_v} on the side of the bipartition to which v and u belong (so that $|M'_{i_v} \cap E(B^*_1)|$ and $|M'_{i_v} \cap E(B^*_2)|$ remain unchanged). See figure 6.2.

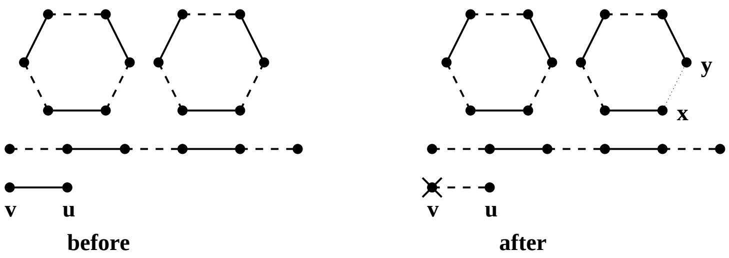

Figure 6.2: M'_i (full) and M'_{i_v} (dashed) before and after, with v marked in M'_i, the any edge reject procedure

second edge reject If there is $w \in B^*$ such that $(u, w) \in M'_{i_v}$, we reject (u, w) and we switch the color of (v, u) from M'_i to M'_{i_v}. See figure 6.3.

We will make sure that no vertex is adjacent to more than $\Delta^{4/5}$ rejected edges by carefully choosing, in every iteration i, the unused matching M'_j we switch with M'_i and the reject procedure. We do this as follows:

3.2.1 If $i > \Delta_1 - 8\Delta^{3/4}$, or if, for some marked vertex v, there is no matching M_{i_v} missing v such that $m(v, i_v) = 0$, we pick any unused M'_j and we use the first edge reject procedure.

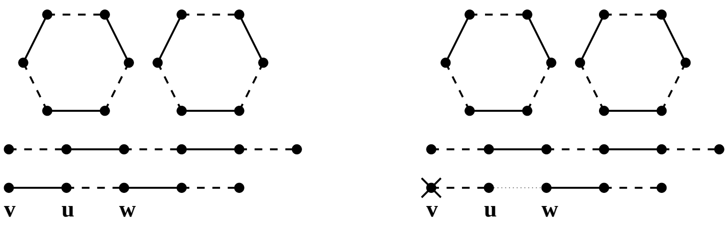

Figure 6.3: M'_i (full) and M'_{i_v} (dashed) before and after, with v marked in M'_i, the second edge reject procedure

3.2.2 If $i < \Delta_1 - 8\Delta^{3/4}$, we choose an unused M'_j such that for every marked vertex v hit by an edge (u, v) of M'_j, either

(i) u is missed by M_{i_v} and, if M_{i_v} is used, $m(u, i_v) = 0$, or

(ii) there is an edge $(u, w) \in M_{i_v}$ and no edge incident to u or w has been rejected in the previous $\frac{1}{4}\Delta^{1/4}$ iterations.

In case **(i)** we use the any edge reject procedure, in which case we choose the "any" edge (x, y) to be rejected so that no edge incident to x or y has been rejected in iterations $i - \lceil\sqrt{\Delta}\rceil, ..., i - 1$. In case **(ii)** we use the second edge reject procedure.

Since no more than than four edges are rejected in each iteration, less than $2\Delta^{1/4}$ edges of M_{i_v} have endpoints incident to an edge rejected in one of the previous $\frac{1}{4}\Delta^{1/4}$ iterations. In addition, if M_{i_v} is used, no more than 4 vertices can be marked in M_{i_v}. So, because the multiplicity of an edge is at most $\sqrt{\Delta}$, it follows that at most $2\Delta^{3/4} + 3\sqrt{\Delta}$ unused matchings have an edge (v, u) where u is either an endpoint of an edge rejected in the previous $\frac{1}{4}\Delta^{1/4}$ iterations or u is marked in M_{i_v}. Because no more than four vertices are marked in M'_i, less than $8\Delta^{3/4}$ unused matchings are not available, so at least one unused matching is available.

No vertex in B^* is incident to more than $8\Delta^{3/4} + 2\delta + \frac{\Delta_1}{\frac{1}{4}\Delta^{1/4}} < \frac{1}{2}\Delta^{4/5}$ rejected edges. The total number of edges rejected is at most $3\Delta_1$. We also note that no more than $2l$ unmarked vertices are missed by M'_i on one side of the bipartition than on the other, for $0 \leq l \leq 4$.

Step 3.3: equalizing the unmarked vertices

We now reject additional edges to insure $n(B_1^*, i) = n(B_2^*, i)$ and, ultimately, $|E(R_1)| = |E(R_2)| < 4\Delta$:

3.3 We reject (up to 4) edges in M_i' to insure that the unmarked big vertices missed by M_i' are split evenly between B_1^* and B_2^*. When picking the edges to reject, we insist that their endpoints have not had an incident edge rejected in the previous $\sqrt{\Delta}$ iterations.

Since there can be at most Δ_1 iterations, no vertex is adjacent to more than $\sqrt{\Delta}$ edges rejected in **3.3**. Furthermore, $|E(R_1)| = |E(R_2)| < 3\Delta_1 + 4\Delta_1 < 4\Delta$ and $\Delta(R_1), \Delta(R_2) < \frac{1}{2}\Delta^{4/5}$.

Finally, we show:

Claim 22 $\delta < n(B_1^*, i) = n(B_2^*, i) < 7\delta$ *for every* $i = 1, ..., \Delta_1$.

Proof: We explicitely made sure that $n(B_1^*, i) = n(B_2^*, i)$ in step **3.3** of iteration i. So we only need to show that $\delta < n(B_1^*, i) < 7\delta$ for any $i = 1, ..., \Delta_1$. We first note that

$$\sum_{i=1}^{\Delta_1} n(B_1^*, i) = \sum_{v \in B_1^*} (\Delta_1 - d_{B_1^*}(v) - m^{\Delta_1}(v)) + 2|R_1| + 2|K_1|$$

Since $\sum_{v \in B_1^*} (\Delta_1 - d_{B_1^*}(v)) \leq \sum_{v \in B_1^*} (\frac{1}{2}\Delta + \delta - \frac{1}{2}d_{B^*}(v) + \frac{1}{8}\delta) \leq \frac{1}{2}\mathrm{def}(B_1^*) + \frac{9}{8}\delta|B_1^*| \leq \frac{3}{4}\Delta + \frac{9}{8}\delta|B_1^*|$, it follows that $n(B_1^*, i) \leq \frac{1}{\Delta_1}(\frac{9}{8}\delta|B_1^*| + 5\Delta) + 8 < 7\delta$.

Similarly, from $\sum_{v \in B_1^*} (\Delta_1 - d_{B_1^*}(v)) \geq \frac{7}{8}\delta|B_1^*|$ and $m^{\Delta_1}(B^*) < \frac{5}{2}\Delta$, it follows that $n(B_1^*, i) \geq \frac{1}{\Delta_1}(\frac{7}{8}\delta|B_1^*| - \frac{5}{2}\Delta) - 8 > \delta$ $\square$

6.2.5 The patching

For $i = 1, ..., \Delta_1$, we recursively obtain M_i by augmenting the vertex disjoint patches we construct between pairs of unmarked big vertices missed by M_i' in $F = G^* - K_1 - R_1 - R_2 - M_1 - ... - M_{i-1}$. After each augmentation, we add the edges of M_i' left uncolored by this augmentation to the reject graphs R_1 or R_2. If we fail to construct a patch between two unmarked vertices missed by some M_i', we will show the existence of and construct a fail pair (X, Y). On the other hand, if we are successfull, the big edges of every matching M_i will miss only the vertices v marked in it (i.e. all v such that $m(v, i) = 1$).

We now describe the construction of the vertex disjoint patches of M_i' in $F = G^* - R_1 - R_2 - M_1 - ... - M_{i-1}$. We recall that a patch P between unmarked vertices $x \in B_1^*$ and $y \in B_2^*$ missed by M_i' is a path from x to y with edges alternating between $E(F) \cap E(B_1^*, B_2^*)$ and $M_i' \cap E(B)$. For $r = 1, ..., n(B_1^*, i)$, we recursively construct the patch P_r as follows:

4.1 We pick a pair of unmarked vertices x_r and y_r on opposite sides of the bipartition so that x_r and y_r are missed by M_i' and have not yet been patched. We pick y_r so it belongs to B. In choosing x_r we always give priority to unmarked, missed vertices in $B^* - B$.

Since $|B^* - B| < 2\Delta^{1/5} < n(B_1^*, i)$, there are more than $|B^* - B|$ vertices in B that are not marked and are missed by M_i' - our choice of the patch endpoints x_r and y_r is thus feasible.

4.2 We define **unavailable** and **usable** vertices. We call $v \in B$ **unavailable** if it is an internal vertex of any patch $P_1, ..., P_{r-1}$ or of any patch constructed for one of the previous $8\lceil\Delta^{1/}\rceil$ matchings $(M_{i-1}, ..., M_I - 8\lceil\Delta^{1/10}\rceil)$. We note that if a vertex is *not* unavailable, then no edge incident to it has been rejected while obtaining one of the previous $8\lceil\Delta^{1/10}\rceil$ matchings. We call $v \in B$ **usable** if $v = y_r$ or $(v, u) \in M_i'$ and neither v nor u is unavailable. If $x_r \in B^* - B$, we also define Y^0-**unavailable** and Y^0**-usable** vertices. We call $v \in B$ Y^0**-unavailable** if it is an internal vertex of any patch $P_1, ...P_{r-1}$, or if it is an endpoint of the second edge of a patch out of a vertex in $B^* - B$ constructed for one of the previous $8\lceil\Delta^{1/10}\rceil$ matchings. We call $v \in B$ is Y^0-usable if $v = y_r$ or $(v, u) \in M_i'$ and neither u nor v are Y^0-unavailable.

4.3 We recursively build the sets X^l and Y^l for $0 \leq l \leq 6\lceil\Delta^{1/20}\rceil$ as follows:

$X^0 = \{x_r\}$,

if $x^r \in B^* - B$, $Y^0 = \{v \in B : v$ is Y^0-usable and $(x_r, v) \in E(F) \cap E(B_1^*, B_2^*)\}$,

if $x^r \in B$, $Y^0 = \{v \in B : v$ is usable and $(x_r, v) \in E(F) \cap E(B_1^*, B_2^*)\}$,

$X^l = \{v \in B : \exists u \in Y^{l-1}$ such that $(u, v) \in M_i'\}$

$Y^l = \{v \in B : v$ is usable and $\exists u \in X^l$ such that $(u,v) \in E(F) \cap E(B_1, B_2)\}$.

If $x_r \in B^* - B$, then at most $6\lceil \Delta^{1/20} \rceil 8\Delta^{1/10}$ vertices in B belong to $P_1, ..., P_{r-1}$ and, as such, are Y^0-unavailable. Furthermore, at most $8\lceil \Delta^{1/10} \rceil$ $|B^* - B| < 8\lceil \Delta^{1/10} \rceil 2\Delta^{1/5}$ vertices are endpoints of the second edge of a patch out of a vertex in $B^* - B$ constructed for one of the previous $8\lceil \Delta^{1/10} \rceil$ matchings, and as such, are Y^0-unavailable. So, at most $\Delta^{2/5}$ vertices are not Y^0-usable.

4.4 If $y_r \in Y^j$ for some $0 \leq j \leq 6\lceil \Delta^{1/20} \rceil$, we construct the patch defined by the sequence of vertices $x_r, y^0, x^1, y^1, ..., y^{j-1}, x^j, y_r$ where $x^l \in X^l$, $y^l \in Y^l$, (x_r, y^0), (x^j, y_r) and (x^l, y^l) belong to $E(F) \cap E(B_1^*, B_2^*)$ and $(y^l, x^{l+1}) \in M_i'$.

Note that each patch contains the same number of edges from $E(B_1)$ and from $E(B_2)$; so when done, $|E(R_1)| = |E(R_2)| < 24\Delta_1\delta\Delta^{1/20} < \frac{1}{10}\Delta^{19/10}$. Furthermore, no vertex is incident to more than $\frac{\Delta_1}{8\Delta^{1/10}} < \frac{1}{8}\Delta^{9/10}$ edges rejected in this step.

4.5 If there is no Y^j containing y_r, then we pick the smallest $j \geq 1$ such that $|Y^j| \leq |X^j| + \frac{1}{2}\Delta^{19/20}$. If $X^j \subset B_1$ then the pair (X, Y), defined by $X = \{v \in X^j \cap B : d^F_{B_2}(v) > \frac{1}{2}\Delta - \Delta^{9/10}\}$ and $Y = N_F(X) \cap B$, form a fail pair.

If $F = G^* - M_1 - ... - M_{i-1}$, then $d^F_{B^*}(v) = \Delta - \Delta_1 - (\mathrm{def}(v) - m^{\Delta_1}(v)) > \frac{1}{2}(\Delta - \Delta_1) > \frac{1}{4}\Delta - \Delta^{9/10}$. Since $\Delta(R_1)$ and $\Delta(R_2)$ are less than $\Delta^{9/10}$, it follows that $d^F_{B_2}(v) > \frac{1}{4}\Delta - \frac{1}{8}\Delta^{19/20}$ for every $v \in B_1^* - v_1$, and similarly, $d^F_{B_1}(v) > \frac{1}{4}\Delta - \frac{1}{8}\Delta^{19/20}$ for every $v \in B_2^*$. Since $d^F_{B^*}(v_1) \geq \Delta(F) - 2$ and because $|K_1| < \frac{1}{4}\Delta$, it follows that $d^F_{B_2}(v_1) > \frac{1}{4}\Delta - \frac{1}{8}\Delta^{19/20}$.

Claim 23 *(X, Y) forms a fail pair in (B_1, B_2).*

Proof: To simplify notation, let $x = x_r \in B_1^*$, so that $y = y_r \in B_2$. Let us first set an upper bound on the sizes of F_1 and F_2, the unavailable vertices in B_1 and B_2, respectively. Since a patch contains at most $6\lceil \Delta^{1/20} \rceil$ vertices in B_1, and since there are at most 8δ patches per matching, it follows that

$$|F_1| < 8\lfloor \Delta^{1/10} \rfloor 8\delta 6 \lceil \Delta^{1/20} \rceil < \frac{1}{8}\Delta^{19/20}$$

for large enough Δ. A symmetrical argument gives $|F_2| < \frac{1}{8}\Delta^{19/20}$.

If $x \in B^* - B$, there are at most $\Delta^{2/5}$ Y^0-unavailable vertices. Since $d^F_{B_2}(x) \geq \frac{1}{4}\Delta - \frac{1}{8}\Delta^{19/20}$ and because the edge multiplicity is at most $\sqrt{\Delta}$, $N^F_{B_2}(x) \geq \frac{1}{5}\sqrt{\Delta}$ and $|X^1| = |Y^0| \geq N^F_{B_2}(x) - \Delta^{2/5} > 0$. So, there must exist a vertex $v \in X^1$, which implies that $|X^2| = |Y^1| \geq d^F_{B_1}(v) - |F_1| > \frac{1}{4}\Delta - \frac{1}{8}\Delta^{19/20}$. If $x \in B$ then $|X^2| = |Y^1| \geq \frac{1}{4}\Delta - \frac{1}{8}\Delta^{19/20}$ easily holds.

Suppose that $|Y^l| > |X^l| + \frac{1}{2}\Delta^{19/20}$ for all $2 \leq l \leq \lceil 6\Delta^{1/20} \rceil - 2$. Then, $|X^{6\lceil \Delta^{1/20} \rceil - 2}| > 3\Delta \geq |B_1|$, a contradiction.

So we must have $|Y^j| \leq |X^j| + \frac{1}{2}\Delta^{19/20}$ for some j between 1 and $\lceil 6\Delta^{1/20} \rceil - 2$, and we pick the minimum j satisfying this property. Let E_1 and E_2 be subsets of B_1 and B_2, respectively, missed by M'_i. Clearly, $|E_k| \leq n(B^*_k, i) + m(B^*_k, i) < 8\delta + 2 < 10\delta$. If j is even (and $X^j \subset B_1$ and $Y^j \subset B_2$), we define $Y = Y^j \cup E_2 \cup F_2$; if j is odd (and $X^j \subset B_2$ and $Y^j \subset B_1$), we define $Y = Y^j \cup E_1 \cup F_1$. Let $X = \{v \in X^j : d^F_Y(v) > \frac{1}{2}\Delta - \Delta^{19/20}\}$. (X, Y) forms a fail pair by the following three properties:

$|Y| \leq |X| + \Delta^{19/20}$.

Proof: We assume $X \subset B_1$ and $Y \subset B_2$; a symmetric argument proves the statement when $X \subset B_2$ and $Y \subset B_1$. We note that $|Y| \leq |Y^j| + |E_2| + |F_2| \leq |X^j| + \frac{1}{2}\Delta^{19/20} + 10\delta + \frac{1}{8}\Delta^{19/20} < |X^j| + \frac{3}{4}\Delta^{19/20}$. Finally, we argue that $|X^j| < |X| + \frac{1}{4}\Delta^{19/12}$ by showing that $X^j - X \subset L$ where $L = v_1 + \{v \in B : \text{def}(v) > 2\sqrt{\Delta}\}$ and $|L| < \sqrt{\Delta}$): if $v \in B_1 - L$ then $d^F_{B_2}(v) \geq d^F_{B^*}(v) - \Delta^{3/5} \geq \Delta(F) - 2\sqrt{\Delta} - \Delta^{3/5} > \frac{1}{2}\Delta - \frac{1}{8}\Delta^{19/20}$, implying $v \notin X^j - X$.

For all $v \in X$: $d_Y(v) > \frac{1}{2}\Delta - \Delta^{19/20}$.

Proof: Assuming $X \subset B_1$, $d_Y(v) > d^F_Y(v) = d^F_{B_2}(v) > \frac{1}{2}\Delta - \frac{1}{8}\Delta^{19/20}$ by the definition of X. A symmetric argument applies if $X \subset B_2$.

If $X \subset B_k$ then $|B_k| - |X| > \frac{1}{4}\Delta - \Delta^{19/20}$.

Proof: If $X \subset B_1$ then y must belong to $B_2 - N^F(X^j)$ which implies that $d^F_X(y) = 0$. Since $y \in B_2$ then $d^F_{B_1}(y) > \frac{1}{4}\Delta - \frac{1}{8}\Delta^{19/20}$. It follows that $|B_1 - X| > \frac{1}{4}\Delta - \frac{1}{8}\Delta^{19}/20$. If $X \subset B_2$, we first note that $|N^F_{B_1}(y)| > \frac{1}{4}\Delta - \frac{1}{8}\Delta^{19/20}$ (since $y \in B_2$). Let Z be the subset of $N^F_{B_2}(y)$ of vertices incident to some edge of $M'_i \cap E(B^1)$. It is easy to check that $N^F_{B_2}(y) - Z$ contains only vertices that are missed by M'_i, that belong to $B^* - B$ or that are forbidden. Since there is fewer than $m(B^*_1, i) + n(B^*_1, i) + |F_1| + |B^* - B| < \Delta^{19/20}$ such vertices, Z must be non-empty. Let (u, v) be an edge of M'_i in $E(B^*_1)$ with $v \in Z$. Since $d^F_{B_2}(u) > \frac{1}{4}\Delta - \frac{1}{8}\Delta^{19/20}$ and $d^F_X(u) = 0$ (because otherwise a patch t y through a vertex in X and vertices u and v would have been possible), $|B_2 - X| > \frac{1}{4}\Delta - \frac{1}{8}\Delta^{19/20}$.

□

6.3 The reject coloring pass

Once we complete the first pass and delete the matchings $M_1, ..., M_{\Delta_1}$ from G^*, we obtain the reduction $F = G^* - \cup_{i=1}^{\Delta_1} M_i = R_1 + H_1 + R_2 + K_1$ where $H_1 \subset E(B^*_1 \cup S^*_1, B^*_2 \cup S^*_2)$, K_1 is a set of $|k_1|$ edges incident to v_1 or v_2 depeneding on whether $k_1 > 0$ or $k_1 < 0$ as defined in 6.4 or in 6.13 and R_1 and R_2 are reject graphs in B^*_1 and B^*_2, respectively, of maximum deggree less than $\Delta^{9/10}$ such that $E(R^*_1)| = |E(R^*_2)| < \frac{1}{8}\Delta^{19/10}$. The deficiency remaining in the graph F at each vertex v is $\text{def}(v) - m^{\Delta_1}(v)$, where $m^{\Delta_1}_u(v) = t_u(v)$ for every $v \in B^*$ and $u = r$ or $u \in S^*$. We will use the fact that the target conditions are satisfied by the marking of the first Δ_1 matchings.

In the reject coloring pass, we will attempt to construct the disjoint matchings $M_{\Delta_1+1}, ..., M_{\Delta_1+\Delta_2}$ in F such that $N = F - \cup_{i=1}^{\Delta_2} M_{\Delta_1+i}$ is a reduction of F (and thus of G^*) such that either N is bipartite, or:

A. $N = H + K$ where $E(H) \subseteq E(B^*_1 \cup S^*_2, B^*_2 \cup S^*_2)$ and K is a subset of edges in K_1.

B. B^*, $B^* - v$ for all $v \in B^*$ and $B^* + u$ for all $u \in S^*$ do not induce an overfull subgraph of maximum degree $\Delta(N)$ in N.

C. $d^N_{B^*}(v) > \frac{1}{2}\Delta(N)$ for every $v \in B^*$, $d^N_{B^*}(u) \leq \frac{1}{2}\Delta(N)$ for every $u \in S^*$ and $d^N_{B^*}(v) + d^N_{B^*-v}(u) \geq \Delta(N)$ for every $u, v \in B^*$.

By lemma 23, if N satisfies these three properties then N does not contain an overfull subgraph of degree Δ, implying that N is $\Delta(N)$ edge colorable.

We will denote $\sum_{i=1}^{\Delta_1+\Delta_2} m(v,i)$ by $m(v)$. To satisfy property **C**, the marking must satisfy:

$$\text{def}(v) - m(v) \leq \frac{1}{2}\Delta(N) \text{ for } v \in B^* \quad (6.18)$$

$$\text{def}_u(B^*) - m_u(B^*) \leq \frac{1}{2}\Delta(N) \text{ for } u \in S^* \quad (6.19)$$

$$\text{def}(v) - m(v) + \text{def}(u) - m(u) + \mu^N(u,v) \leq \frac{1}{2}\Delta(N) \text{ for } u,v \in B^* \quad (6.20)$$

We call $v \in B^*$ **bad** if $\text{def}(v) \geq \frac{1}{2}\Delta(N) = \frac{1}{2}(\Delta - \Delta_1 - \Delta_2)$, and we call $u \in S^*$ **bad** if $\text{def}_u(B^*) > \Delta - \Delta_1 - \frac{1}{4}\delta$. Note that if $u \in S^*$ is not bad, then the remaining deficiency in F induced by u, $\text{def}_u(B^*) - m_u(B^*)$, is no greater than $\frac{1}{2}(\Delta - \Delta_1)$. We observe that, since the marking of the first Δ_1 matchings satisfies target conditions 6.5 - 6.7, the properties 6.18 - 6.20 will hold if we mark every bad big vertex in at least $\frac{1}{2}\Delta_2$ of the last Δ_2 matchings and we insure that every bad small vertex induces a mark in at least $\frac{1}{2}\Delta_2$ of the last Δ_2 matchings.

Property **B** will hold in the odd $|B^*|$ case if and only if

$$\text{def}(B^*) - m(B^*) \geq \Delta(N) \quad (6.21)$$

and in the even $|B^*|$ case if and only if

$$2(\text{def}(v) - m(v)) \leq \text{def}(B^*) - m(B^*) \text{ for all } v \in B^* \quad (6.22)$$

$$2(\text{def}_u(B^*) - m_u(B^*)) \leq \text{def}(B^*) - m(B^*) \text{ for all } u \in S^* \quad (6.23)$$

Let n_b and n_s be the numbers of bad big vertices and bad small vertices, respectively. In this, smallest deficiency, case, n_b is at most 10 and the only candidates for bad big vertices are the 10 smallest degree vertices $b_1, b_3, ..., b_9$ in B_1^* and $b_2, b_4, ..., b_{10}$ in B_2^* such that $\text{def}(b_1) \geq \text{def}(b_2) \geq ... \geq \text{def}(b_{10})$. Let c be the smallest even integer greater than $\frac{3}{4}\Delta_2$. We obtain the set K by removing m edges from K_1, where m depends on the number of bad vertices. If $|B^*|$ is odd, then $m = 0$ if $k_1 \leq \frac{1}{16}\Delta$ or n_b, n_s are both odd; if $k_1 > \frac{1}{16}\Delta$, then $m = \frac{1}{2}(\Delta_2 - c)$ if $n_b + n_s$ is odd and $m = \frac{1}{2}\Delta_2$ if n_b, n_s are both even.

We construct the matchings $M_{\Delta_1+1}, ..., M_{\Delta_1+\Delta_2}$ covering $E(R_1) \cup E(R_2) \cup (K_1 - K)$ in three steps.

6.3.1 The initial coloring

We first construct an initial coloring:

1. We construct initial matchings $M'_1, ..., M'_{\frac{1}{2}\Delta_2}$ balanced in B_1^* and in B_2^* and covering $R_1 \cup R_2 \cup (K_1 - K)$ using Fournier's edge coloring algorithm and our balancing procedure (see 3.2.2). We insist that the last m matchings have one more edge in B_1^* than in B_2^* and that the first $\frac{1}{2}c$ matchings miss all big bad vertices.

Since $\Delta_2 \geq \frac{1}{2}\Delta^{19/20}$ and $|E(R_1)| = |E(R_2)| < \frac{1}{8}\Delta^{19/20}$, it follows that $|M'_i \cap E(B_1^*)|$ and $|M'_i \cap E(B_2^*)|$ are less than $\lceil \frac{1}{4}\Delta^{19/20} \rceil$.

6.3.2 The marking

We use the initial coloring to construct disjoint matchings $M''_{\Delta_1+1}, ..., M''_{\Delta_1+\Delta_2}$ covering $E(R_1) \cup E(R_2) \cup (K_1 - K)$ and to define a proper marking over those matchings such that, for every $i = 1, ..., \Delta_2$:

(i) $|M_{\Delta_1+i} \cap E(B_1^*)|$ and $|M_{\Delta_1+i} \cap E(B_1^*)|$ are less than $\lceil \frac{1}{4}\Delta^{19/20} \rceil$,

(iii) $m(B^*, \Delta_1 + i) \leq 10$, and

(ii) $n(B_1, \Delta_1 + i) = n(B_1^*, \Delta_1 + i) = n(B_2^*, \Delta_1 + i) = n(B_2, \Delta_1 + i)$.

The second condition implies that in no maching M''_{Δ_1+i} a vertex v in $B^* - B$ is missed and not marked. So, we will only need to patch vertices in B in the patching step. We define the marking and we construct the matchings by repeating the following for $i = 1, 2, ..., \frac{1}{2}\Delta_2$:

2.1 We define $m_r(v, \Delta_1 + 2i - 1)$, $m_u(v, \Delta_1 + 2i - 1)$, $m_r(v, \Delta_1 + 2i)$, and $m_u(v, \Delta_1 + 2i)$ for every $v \in B^*$ and $u \in S^*$.

2.2 We construct disjoint matchings M''_{Δ_1+2i-1} and M_{Δ_1+2i} in $F' = F - M''_{\Delta_1+1} - M''_{\Delta_1+2} - ...M''_{\Delta_1+2i-2}$ so their union contains M'_i.

We describe the details of each step in the case when $|B^*|$ is odd first.

Step 2.1: The marking in iteration i for odd $|B^*|$

We first define the marking in iteration $i \le \frac{1}{2}\Delta_2$ in the case when there is no bad vertex (i.e. $n_b + n_S = 0$):

2.1.1 If $k_1 < \frac{1}{16}\Delta$ (in which case $m = 0$), we set $m_{s_1}(v_1, \Delta_1 + 2i - 1) = 1$ for some $v_1 \in B_1^*$ with $\mathrm{def}_{s_1}(v_1) > m_{s_1}^{\Delta_1+2i-2}(v_1)$; let $W_1 = \{v_1\}$.

Then, we set $m_{s_1}(v_2, \Delta_1 + 2i) = 1$ for some $v_2 \in B_1^* - v_1$ with $\mathrm{def}_{s_1}(v_2) > m_{s_1}^{\Delta_1+2i-1}(v_2)$ and $(v_1, v_2) \notin M_i'$. Let $W_2 = \{v_2\}$.

2.1.2 If $k_1 \ge \frac{1}{16}\Delta$ (in which case $m = \frac{1}{2}\Delta_2$), we set $m_u(v_1, \Delta_1 + 2i - 1) = 1$ for some $v \in B_2^*$ with $\mathrm{def}_u(v_1) > m_u^{\Delta_1+2i-2}(v_1)$ where $u = r$ or $u = s_2$. Let $W_1 = \{v_1\}$.

Then, we set $m_u(v_2, \Delta_1 + 2i) = 1$ for some $v_2 \in B_1^*$ with $\mathrm{def}_u(v_2) > m_u^{\Delta_1+2i-1}(v_2)$ where $u = r$, $u = s_1$ or $u = s_3$. Let $W_2 = \{v_2\}$. (We illustrate the marking in figure 6.4.)

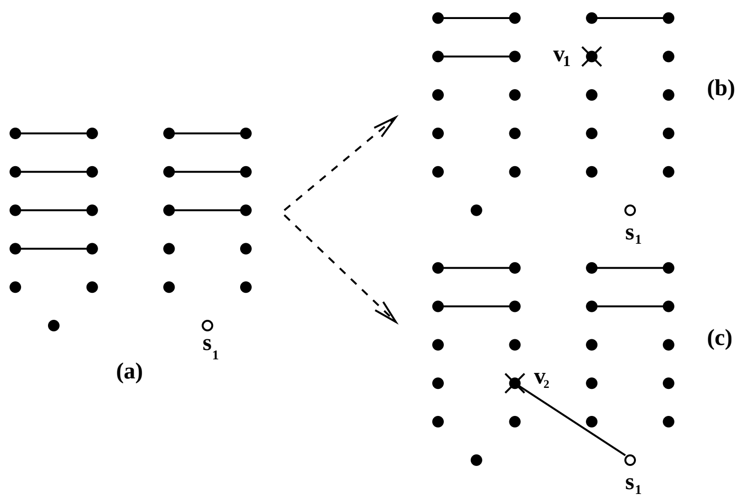

Figure 6.4: 2.1.2: M_i' (a) and the marking in M''_{2i-1} (b) and M''_{2i} (c)

In step **2.1.2**, the remaining deficiency in B_1^* and B_2^* is at least $\Delta_1 - k_1 > \frac{1}{4}\Delta - \Delta^{9/10}$ and $k_1 \ge \frac{1}{2}\Delta_2$, respectively.

If $n_b + n_S > 0$, we need to define a marking in which all bad big vertices are marked at least $\frac{1}{2}\Delta_2$ times and all bad small vertices induce marks at least

$\frac{1}{2}\Delta_2$ times. To do this we define the marking in every iteration $i \leq \frac{1}{2}c$ in one of the following ways:

2.1.3 If $n_b + n_s$ is odd, we set $m_r(v, \Delta_1 + 2i - 1) = 1$ for every bad $v \in B^*$. We then set $m_u(v, \Delta_1 + 2i - 1) = 1$ for every bad $u \in S^*$ where $v \in B^*$ is chosen so that $\text{def}_u(v) > m_u^{\Delta_1+2i-2} > 0$; let W_1 be the set of big vertices v such that $m(v, \Delta_1 + 2i - 1) = 1$.

Then, we set $m_r(v, \Delta_1 + 2i) = 1$ for ever bad $v \in B^*$ and $m_u(v, \Delta_1 + 2i) = 1$ for every bad $u \in S^*$ where $v \in B^*$ is chosen so that $\text{def}_u(v) > m_u^{\Delta_1+2i-1}$ and there is no $u \in W_1$ with $(v, u) \in M_i'$. Let W_2 be the set of big vertices v such that $m(v, \Delta_1 + 2i) = 1$.

2.1.4 If n_b and n_s are both odd (in which case $m = 0$), we first pick the two least recently chosen vertices v_1, v_2 among the bad vertices in $B_1^* \cup S_2^*$. (For example, $v_1, v_2 = b_1, s_1$ if $n_b = n_S = 1$.) We set $m_r(v, \Delta_1 + 2i - 1) = 1$ for every bad $v \in B^* - v_1$ and $m_u(v, \Delta_1 + 2i - 1) = 1$ for every bad $u \in S^* - v_1$ where $v \in B^*$ is chosen so that $\text{def}_u(v) > m_u^{\Delta_1+2i-2}(v)$. Let W_1 be the set of big vertices v such that $m(v, \Delta_1 + 2i - 1) = 1$.

Then, we set $m_r(v, \Delta_1 + 2i) = 1$ for every bad $v \in B^* - v_2$ and $m_u(v, \Delta_1 + 2i) = 1$ for every bad $u \in S^* - v_2$ where $v \in B^*$ is chosen so that $\text{def}_u(v) > m_u^{\Delta_1+2i-1}(v)$ and there is no $u \in W_1$ with $(v, u) \in M_i'$. Let W_2 be the set of big vertices v such that $m(v, \Delta_1 + 2i) = 1$. (We illustrate the marking when $n_b = n_s = 1$ in figure 6.5.)

2.1.5 If n_b and n_s are both even and $k_1 < \frac{1}{16}\Delta$ (in which case $m = 0$), we do as in **2.1.3**, with the following addition: if s_1 is not bad, we add it to the bad vertices, otherwise we add s_3.

2.1.6 If n_b and n_s are both even and $k_1 > \frac{1}{16}\Delta$ (in which case $m = \frac{1}{2}\Delta_2$, we first pick a least recently chosen bad vertices $v_1 \in B_1^* \cup S_2^*$ and $v_2 \in B_2^* \cup S_1^*$. We set $m_r(v, \Delta_1 + 2i - 1) = 1$ for every bad $v \in B^* - v_1$ and we set $m_u(v, \Delta_1 + 2i - 1) = 1$ for every bad $u \in S^* - v_1$ where $v \in B^*$ is chosen so that $\text{def}_u(v) > m_u^{\Delta_1+2i-2}(v)$. Let W_1 be the set of vertices such that $m(v, \Delta_1 + 2i - 1) = 1$.

Then, we set $m_r(v, \Delta_1 + 2i) = 1$ for every bad $v \in B^* - v_2$ and we set $m_u(v, \Delta_1 + 2i) = 1$ for every bad $u \in S^* - v_2$ where $v \in B^*$ is chosen so that $\text{def}_u(v) > m_u^{\Delta_1+2i-1}(v)$ and there is no $u \in W_1$ with $(v, u) \in M_i'$. Let W_2 be the set of vertices such that $m(v, \Delta_1 + 2i) = 1$.

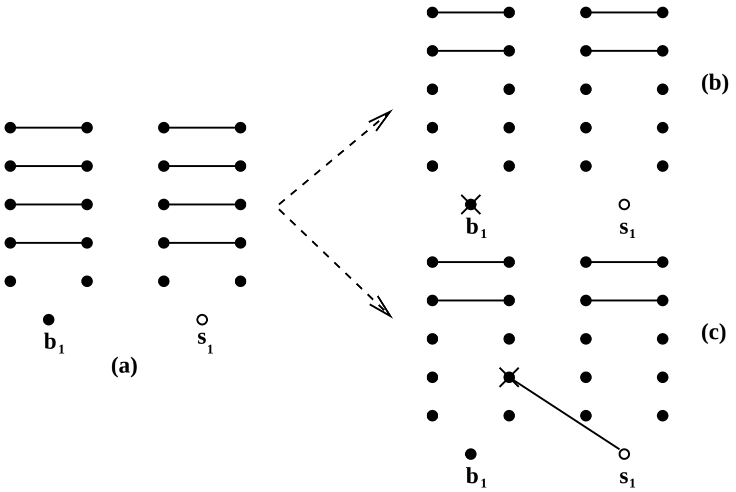

Figure 6.5: 2.1.4: M_i' (a) and the marking in M''_{2i-1} (b) and M''_{2i} (c)

In all iterations $i > \frac{1}{2}c$, we define the marking exactly as in **2.1.1** or **2.1.2**, except if $n_b = n_S = 1$ in which case we just keep doing **2.1.4**.

Step 2.1: The marking when $|B^*|$ is even

We define the marking in one of several ways, depending on the number of bad vertices. If the total number of bad vertices, $n_b + n_s$, is even, then $m = 0$, all initial matchings have an equal number of edges in B_1^* and B_2^* and we define the marking in iteration $i \leq \frac{1}{2}c$ as follows:

2.1.1 If $n_b + n_s$ is even, we set $m_r(v, \Delta_1 + 2i - 1) = 1$ for every bad big v and, for every bad $u \in S^*$, we choose a different, unmarked and non-bad vertex $w \in B^*$ such that $\mathrm{def}_u(v) > m_u^{\Delta_1+2i-2}(v)$ and we set $m_u(w, \Delta_1 + 2i - 1) = 1$. Let $W_1 = \{v \in B^* : m(v, \Delta_1 + 2i - 1) = 1$. (Part 1)

Then, we set $m_r(v, \Delta_1+2i) = 1$ for every bad $v \in B^*$ and, for every bad $u \in S^*$, we choose a different, unmarked and non-bad $w \in B^*$ such that $\mathrm{def}_u(w) > m_u^{\Delta_1+2i-1}(w)$ and there is no $w' \in W_1$ with $(w, w') \in M_i'$ and we set $m_u(w, \Delta_1+2i) = 1$. Let $W_2 = \{v \in B^* : m(v, \Delta_1+2i-1) = 1$. (Part 2) (We illustrate the marking when $n_b = n_s = 1$ in figure 6.5.)

We put no marks in iterations $i = \frac{1}{2}c + 1, ..., \frac{1}{2}\Delta_2$.

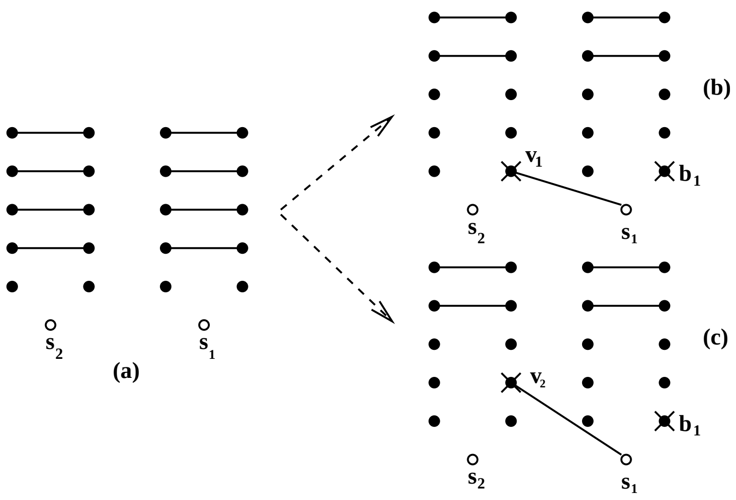

Figure 6.6: 2.1.1: M_i' (a) and the marking in M''_{2i-1} (b) and M''_{2i} (c)

If $n_b + n_s$ is odd and at least 5, then $m = 0$ and we define the marking in iteration $i \leq \frac{1}{2}c$ as follows:

2.1.2 We pick the two least recently chosen two bad vertices v_1, v_2 in $B_2^* \cup S_1^*$ if n_S is even, or in $B_1^* \cup S_2^*$ if n_S is odd. We then define the marking as in **2.1.1** except that we do not use v_1 in the first part, and v_2 in the second part.

We put no marks in iterations $i = \frac{1}{2}c + 1, ..., \frac{1}{2}\Delta_2$.

If $n_b + n_s = 1$ or $n_b + n_s = 3$, we need to be more careful in defining our marking. We consider first the cases when n_s is even and $k_1 < \frac{1}{16}\Delta$, or n_s is odd and $k_1 > -\frac{1}{16}\Delta$. In both cases we defined $m = 0$, and so all initial matchings have the same number of edges in B_1^* and B_2^*. If n_s is even and $k_1 < \frac{1}{16}\Delta$, we define the marking in iteration $i \leq \frac{1}{2}c$ as follows:

2.2.3 We do as in **2.1.1** and, in addition, we set $m_u(v, \Delta_1 + 2i - 1) = 1$ in Part 1, for some non-bad vertex $v \in B_1^*$ and for $u = r$ or for non-bad $u \in S_2^*$ such that $\mathrm{def}_u(v) > m_u^{\Delta_1+2i-2}(v)$, and we set $m_u(v, \Delta_1{+}2i) = 1$ in Part 2, for some non-bad $v \in B_1^*$ and for $u = r$ or for non-bad $u \in S^*$ such that $\mathrm{def}_u(v) > m_u^{\Delta_1+2i-2}(v)$.

We put no marks in iterations $i = \frac{1}{2}c+1, ..., \frac{1}{2}\Delta_2$. We can choose a vertex v as desired, in parts 1 and 2, because there is at least $\frac{1}{8}\Delta$ available deficiency in B_1^* that is not on bad big vertices or induced by bad small vertices, which follows from $(\text{def}(B_2^*) - m^{\Delta_1}(B_2^*)) - (\text{def}(B_1^*) - m^{\Delta_1}(B_1^*)) = 2k_1 < \frac{1}{8}\Delta$ and the fact that there is more than $\frac{1}{4}\Delta - \delta$ more bad available deficiency in B_2^* and B_1^*. We omit the case when n_s is odd and $k_1 > -\frac{1}{16}\Delta$ as the marking is symmetric to **2.2.3**.

If $n_b + n_s = 3$, n_s is even and $k_1 > \frac{1}{16}\Delta$, then $m = \frac{1}{3}c$ and the first $\frac{1}{3}c$ initial matchings have one more edge in B_1^* than in B_2^*. The two possible cases are when b_1, b_2, b_3 are bad and when s_2, s_1, b_1 are bad. We omit the second case as it is similar to the first, for which we define the marking as follows:

2.2.4 If $i \leq \frac{1}{3}c$, we set $m_r(b_1, \Delta_1 + 2i - 1) = m_r(b_3, \Delta_1 + 2i - 1) = 1$ and let $W_1 = \{b_1, b_3\}$. Then we set $m_r(b_1, \Delta_1 + 2i) = m_r(b_2, \Delta_1 + 2i) = 1$ and let $W_2 = \{b_1, b_2\}$.

If $\frac{1}{3}c+1 \leq i \leq \frac{1}{2}c$, we set $m_r(b_2, \Delta_1 + 2i - 1) = m_r(b_3, \Delta_1 + 2i - 1) = 1$ and let $W_1 = \{b_2, b_3\}$. Then we set $m_r(b_2, \Delta_1 + 2i) = m_r(b_3, \Delta_1 + 2i) = 1$ and let $W_2 = \{b_2, b_3\}$.

We put no marks in iterations $i = \frac{1}{2}c+1, ..., \frac{1}{2}\Delta_2$. Note that each bad vertex is marked $\frac{2}{3}c \geq \frac{1}{2}\Delta_2$ times. If $n_b + n_s = 3$, n_s is odd and $k_1 < -\frac{1}{16}\Delta$, then $m = -\frac{1}{3}c$ and the first $\frac{1}{3}c$ initial matchings have one more edge in B_2^* than in B_1^*. The two possible cases are when b_1, b_2, b_3 are bad and when s_2, s_1, b_1 are bad. We omit the definition of the marking in this case as it symmetric to **2.2.4**.

If $n_b = 1$ and $k_1 > \frac{1}{16}\Delta$ then $m = \frac{1}{2}\Delta_2$, all initial matchings have one more edge in B_1^* than in B_2^* and we define the marking in iteration $i \leq \frac{1}{2}\Delta_2$ as follows:

2.1.2 We set $m_r(b_1, \Delta_1 + 2i - 1) = m_u(v, \Delta_1 + 2i - 1) = 1$ for some $v \in B_2^* - b_1$ with $\text{def}_u(v) > m_u^{\Delta_1 + 2i - 2}(v)$ where $u = r$ or $u = s_1$. Let $W_1 = \{b_1, v\}$.

Note that $m(v, \Delta_1 + 2i) = 0$ for all $v \in B^*$. We can choose v with available deficiency because $\text{def}(B_2^* - b_1) - m^{\Delta_1}(B_2^* - b_1) > k_1 > \frac{1}{16}\Delta$, which follows from target conditions 6.13 and 6.14. If s_1 is bad and $k_1 < -\frac{1}{16}\Delta$, then $m = -\frac{1}{2}\Delta_2$ and all Δ_2 initial matchings have one more edge in B_2^* than in B_1^* and we define the marking as follows:

2.1.3 We set $m_{s_1}(v_1, \Delta_1+2i-1) = m_u(v_2, \Delta_1+2i-1) = 1$ for some different vertices $v_1, v_2 \in B_1^*$ with $\mathrm{def}_{s_1}(v_1) > m_{s_1}^{\Delta_1+2i-2}(v_1)$ and $\mathrm{def}_u(v_2) > m_u^{\Delta_1+2i-2}(v_2)$ where $u = r$ or $u = s_3$.

Again, $m(v, \Delta_1 + 2i) = 0$ for all $v \in B^*$. We can choose v_1 because $\mathrm{def}_{s_1}(B_1^*) - m^{\Delta_1}(B_1^*) > \frac{1}{4}\Delta - \delta$. We can choose v_2 because there is at least $k_1 > \frac{1}{16}\Delta$ available deficiency in B_1^* that is *not* induced by s_1, which follows from target conditions 6.13 and 6.15.

Step 2.2: constructing M_{Δ_1+2i-1} and M_{Δ_1+2i}

2.2.1 We remove from M_i' all edges incident to vertices v such that $m(v, \Delta_1 + 2i-1) = 1$, and we add them to M''_{Δ_1+2i}. Since none of these edges belong to (W_1, W_2), they miss all vertices v such that $m(v, \Delta_1 + 2i - 1) = 1$. Let $W = W_1 + W_2 + \{v \in B^* : (u, v) \in M_i' \text{ and } v \in W_2\}$ and note that $|W| \leq 21$.

2.2.2 We now construct M''_{2i-1} from M_i' so that no vertex in $B^* - B$ is missed and unmarked. Let U be the set of unmarked vertices in $B^* - B$ missed by M_i'. We construct a matching M in $(F - W) \cap (B_1^*, B_2^*) - M''_{\Delta_1+1} - ... - M''_{\Delta_1+2i-2}$ such that every $v \in U$ is an endpoint of some edge in M. (We esily obtain this matching because the neighborhood of every $v \in B^* - B$ is at least $\frac{1}{3}\sqrt{\Delta} > |B^* - B|$.) We remove from M_i' edges incident to M and we add them to M''_{Δ_1+2i}. We remove additional edges not incident to vertices in W_2 from $M_i' \cap B_1^*$ or $M_i' \cap B_2^*$ and we add them to M''_{Δ_1+2i} so $|M''_{\Delta_1+2i} \cap B_1^*| = |M''_{\Delta_1+2i} \cap B_2^*| < \Delta^{1/5}$. We then set $M''_{\Delta_1+2i-1} = M_i' \cup M$.

2.2.3 We now finish the construction of M''_{Δ_1+2i}, again so that no vertex in $B^* - B$ is missed and unmarked. Let U be the set of unmarked vertices in $B^* - B$ missed by M''_{Δ_1+2i}. Let X be the set of big vertices that are endpoints of edges in M''_{Δ_1+2i}. We construct a matching M in $(F - W_2 - X) \cap (B_1^*, B_2^*) - M''_{\Delta_1+1} - ... - M''_{\Delta_1+2i-1}$ such that every $v \in U$ is an endpoint of some edge in M. We add M to the final M''_{Δ_1+2i}.

6.3.3 The patching

We now attempt to construct the matchings $M_{\Delta_1+1}, ..., M_{\Delta_1+\Delta_2}$. For $i = 1, ..., \Delta_2$, we obtain M_{Δ_1+i} by augmenting M''_{Δ_1+i} in $H = F - M_{\Delta_1+1} - ... -$

M_{Δ_1+i-1} with a disjoint matching hitting all unmarked vertices in B missed by M''_{Δ_1+i} as follows:

3. Let U_1 and U_2 be the sets of unmarked vertices missed by M''_{Δ_1+i} in B_1 and B_2, respectively. We attempt to find a perfect matching M in the bipartite subgraph defined by the bipartition (U_1, U_2) and with edge set $E(H) \cap E(U_1, U_2)$.

If we successfully obtain such a matching M, we add its edges to M''_{Δ_1+i} to obtain M_{Δ_1+i}.

If we fail to obtain M, we find the sets $X' \subset U_1$ and $Y' = N^H_{U_2}(X')$ such that $|X'| > |Y'|$ and we set $X'' = U_2 - Y'$ and $Y'' = U_1 - X' = N^H_{U_1}(X'')$. Let $Y = B_2 - X''$ and $X = \{v \in X' : d^H_{B_2}(v) > \frac{1}{2}\Delta - \Delta^{19/20}\}$.

Claim 24 *(X, Y) forms a fail pair in (B_1, B_2).*

Proof: The claim is true if the following three properties hold:

i. $|Y| < |X| + \Delta^{19/20}$.
Proof: $|Y| \leq |Y'| + 2|M''_{\Delta_1+i}| + m(B_2, \Delta_1 + i) \leq |X'| + \frac{1}{2}\Delta^{9/10} + 2 < |X| + \Delta^{9/10}$. The last inequality follows from $X' - X \subset L$ where $L = \{v \in B : \mathrm{def}(v) > 2\sqrt{\Delta}\}$ is of cardinality $2\sqrt{\Delta}$: since if $v \in B_1 - L$ then $d^H_{B_2}(v) > d^F_{B_2}(v) - i > \frac{1}{2}\Delta - 3\Delta^{9/10} - \frac{1}{2}\Delta^{19/20} < \frac{1}{2}\Delta - \Delta^{19/20}$, and if $v \in B_2 - L$, $d^H_{B_1}(v) > \frac{1}{2}\Delta - \Delta^{19/20}$.

ii. For all v in X: $d_Y(v) > \frac{1}{2}\Delta - \Delta^{19/20}$.
Proof: $d_Y(v) \geq d^H_Y(v) = d^H_{B_2}(v) > \frac{1}{2}\Delta - \Delta^{19/20}$ by definition of X.

iii. $|B_1 - X| > \frac{1}{4}\Delta - \Delta^{19/20}$.
Proof: Since $|X''| > |Y''|$, X'' is not empty. If $v \in X''$ then $d^H_X(v) = 0$. It follwos that $|B_1 - X| > d^H_{B_1}(v) > d^H_{B_1^*}(v) - \Delta^{3/5} > \frac{1}{4}\Delta - \Delta^{19/20}$.

□

Appendix A

Near-bipartite graphs

For completeness, we include the proof by Reed [Ree95] that $\chi'(G) = \lceil\chi^*(G)\rceil$ for any near-bipartite graph G. Actually, he shows that we can determine whether G has an overfull subgraph using just one application of the max. flow - min.cut algorithm. With a bit more work, we can obtain an edge coloring of G using $\lceil\chi^*(G)\rceil$ colors.

Consider a graph G and a vertex v of G such that $G - v$ is bipartite with bipartition (A, B). Let k be any integer greater than or equal to Δ. We present an algorithm for determining if G has an edge coloring using k matchings. If the algorithm does not return a k edge coloring of G, it returns a subgraph H of G such that $|E(H)| > k\lceil\frac{|H|-1}{2}\rceil$. This shows that $\chi^*(G) > k$.

The algorithm relies on the following

Key Observation: *Let G be a near-bipartite graph. Let v be a vertex of G such that $G - v$ is bipartite with bipartition (A, B). Let $k \geq \Delta(G)$ be an integer. Let $l = d_A(v)$. Then, there is an k edge coloring of G if and only if there exist a partition of G into two subgraphs G_1 and G_2 such that $\Delta(G_1) = l, \Delta(G_2) \leq k - l$, and G_1 contains all the edges between v and A.*

Proof: If G has a edge coloring using k matchings then defining G_1 to be the set of matchings containing an edge between v and A, and setting $G_2 = G - G_1$ yields the desired partition.

Conversely, note that for any such partition into G_1 and G_2, both G_1 and G_2 are bipartite. Thus, G_1 has an edge coloring using $l = \Delta(G_1)$ colors and G_2 has an edge coloring using $\Delta(H_2) \leq k - l$ matchings. So, G has an edge coloring using k matchings. $\square$

This key observation implies that rather than trying to find a k edge coloring of G directly, we can simply test for the existence of an appropriate partition of $E(G)$ into two subgraphs G_1 and G_2. Actually, we will try to construct $G_1 - v$. So, we let l be the number of edges between x and A. We let m be the number of edges between x and B. By symmetry, we can assume that $l \leq m$. We attempt to find a subgraph F of $G - v$ satisfying:

(1) $\forall w \in A, d_{G-v}(w) - (k - l) \leq d_F(w) \leq l - \mu(v, w)$, and

(2) $\forall w \in B, d_G(w) - (k - l) \leq d_F(w) \leq l$.

To determine if such a subgraph exists, we solve a maximum flow problem on a network G^* obtained from $G - v$ as follows:

(i) directing all edges from A to B,

(ii) adding a new node *dummy*, an arc from *dummy* to each node of B, and an arc to dummy from each node of A,

(iii) giving a capacity of one to each arc corresponding to an edge of G,

(iv) giving a capacity of $k - d_G(w)$ to the arc between w and *dummy*,

(v) setting the demand at each node of B to l, and

(vi) setting the demand at each node w of A to $-(l - \mu(v, w))$, and

(vii) setting the demand at *dummy* to $l(|A| - |B| - 1)$.

Now, if the desired flow exists then an integer valued flow exists and can be found in $O(|E(G)|V(G)|log(|V(G)|))$ time using standard network flow techniques (see [LP86]). The edges of G with flow one clearly yield the desired F and hence the partition H_1 and H_2.

If the desired flow does not exist then there is a set $S \subseteq V(G^*)$ such that $\sum_{v \in S} demand(v) - \sum_{uv \in E(G^*), v \in S, u \in V-S} capacity(uv) > 0$. We show now how to convert this cut into the desired overfull subgraph.

Case 1: *dummy* $\notin S$

Let $A' = A \cap S, B' = B \cap S$. We have:

$l|B'| - l|A'| + \sum_{a \in A'} \mu(v, a) - \sum_{b \in B'} (k - d_G(b)) - \sum_{a \in A'} \sum b \in B' \mu(a, b) > 0.$

or equivalently with H the subgraph induced by $S + v$:

$$|E(H)| - k|B'| + l(|B'| - |A'|) > 0 \tag{A.1}$$

Now, clearly $|E(H)| \leq \Delta|B'| + l \leq k|B'| + l$ so $|B'| \geq |A'|$. On the other hand, $|E(H)| \leq \Delta|A'| + (\Delta - l) \leq k|A'| + (k - l)$. So by A.1, we must have $|B'| \leq |A'|$. So $|B'| = |A'|$, and we obtain: $|E(H)| > k|B'| = \frac{k(|H|-1)}{2}$. Now, H is the desired overfull subgraph.

Case 2: *dummy* $\in S$

Let $A' = A - S, B' = B - S, S' = A' \cup B'$. We know that the sum of the demands in S' added to the capacity of the arcs from S' to S is negative.

Thus,

$$l|B'| - l|A'| + \sum_{a \in A'} \mu(v,a) + \sum_{a \in A'} (k - d_G(a)) + \sum_{a \in A'} \sum_{b \in B - B'} \mu(a,b) < 0$$

or equivalently with H the subgraph induced by S':

$$|E(H)| > k|A'| + l(|B'| - |A'|)$$

But this is impossible as $|E(H)| \leq k|A'|$, $|E(H)| \leq k|B'|$, and $l \leq k$.

Thus, we see that the chromatic index of G is indeed the roundup of its fractional chromatic index. We also know that the chromatic index of G is at most $\frac{3\Delta}{2}$ because we can delete $\frac{\Delta}{2}$ of the edges from v and obtain a bipartite graph. Thus, applying the procedure described above $\frac{\Delta}{2}$ times allows us to determine the chromatic index of G. The algorithm also returns two bipartite graphs H_1 and H_2 such that $\chi'(G) = \Delta(H_1) + \Delta(H_2)$. We can use a standard algorithm[LP86] which runs in $O(\Delta^2|E(G)|)$ time to color each of these graphs and thereby obtain a coloring of G. Actually, by using binary search we can determine $\chi'(G)$ using at most $log(\Delta)$ calls to the above procedure. Thus, we obtain an $O((\Delta^3 + \Delta^2(log\Delta)(log|V(G)|))|V(G)|^2)$ time algorithm for edge coloring nearly bipartite graphs of maximum degree Δ.

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